

EXTENDED OBSTRUCTION TENSORS AND RENORMALIZED VOLUME COEFFICIENTS

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1. INTRODUCTION

In recent years there has been a great deal of progress on the so-called σ_k -Yamabe problem. In [CF], Alice Chang and Hao Fang have suggested that a variant of this problem might also be fruitful to study. The main goal of this paper is to investigate the algebraic structure under conformal transformation of the renormalized volume coefficients, the curvature quantities considered by Chang-Fang. A key ingredient in the investigation is the introduction of “extended obstruction tensors”, which are anticipated to be of independent interest. These are natural tensors associated to a pseudo-Riemannian metric g which turn out to be building blocks for the expansion of the ambient or Poincaré metric determined by g and thus also for the renormalized volume coefficients.

The σ_k -Yamabe problem was introduced by Jeff Viaclovsky in [V]. Let

$$P_{ij} = \frac{1}{n-2} \left(R_{ij} - \frac{R}{2(n-1)} g_{ij} \right)$$

denote the Schouten tensor of a metric g on a manifold M of dimension $n \geq 3$, and let $g^{-1}P$ denote the endomorphism P^i_j obtained by raising an index. For $1 \leq k \leq n$, the σ_k -Yamabe problem is to find a metric in a given conformal class for which $\sigma_k(g^{-1}P)$ is constant, where $\sigma_k(A)$ denotes the k -th elementary symmetric function of the eigenvalues of an endomorphism A . We set $\sigma_k = 0$ for $k > n$. For $k = 1$, $\sigma_1(g^{-1}P)$ is a multiple of the scalar curvature of g , so this is the Yamabe problem. For $2 \leq k \leq n$, $\sigma_k(g^{-1}P)$ is a second order fully nonlinear operator in the conformal factor.

Variational methods have played an important role in the study of this problem. In dimensions $n > 2$, the Yamabe equation $R = c$ is the Euler-Lagrange equation for the total scalar curvature functional $\int_M R dv_g$ under conformal variations subject to the constraint $\text{Vol}_g(M) = 1$. Of course, this fails when $n = 2$ because of the Gauss-Bonnet Theorem. For $k = 2$ the analogous special dimension is $n = 4$. In this dimension, the total σ_2 curvature $\int_M \sigma_2(g^{-1}P) dv_g$ is a conformal invariant. If $n \neq 4$, the natural generalization of the variational characterization of the Yamabe equation holds: the equation $\sigma_2(g^{-1}P) = c$ is the Euler-Lagrange equation for the

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functional $\int_M \sigma_2(g^{-1}P) dv_g$ under conformal variations subject to the constraint $\text{Vol}_g(M) = 1$.

Both of these properties fail for general metrics when $3 \leq k \leq n$. The special dimension is now $n = 2k$. But it is no longer true that $\int_M \sigma_k(g^{-1}P) dv_g$ is a conformal invariant in dimension $2k$. Nor is it true for $n \neq 2k$ that the equation $\sigma_k(g^{-1}P) = c$ is the constrained Euler-Lagrange equation for the total σ_k functional. Viaclovsky did show that both properties hold if g is locally conformally flat. But Branson and Gover proved in [BG] that if $3 \leq k \leq n$ and g is not locally conformally flat, then the equation $\sigma_k(g^{-1}P) = c$ is not the Euler-Lagrange equation of any functional.

The renormalized volume coefficients of g , denoted here by $v_k(g)$, arose in the late '90's in the physics literature in the context of the AdS/CFT correspondence. A mathematical discussion is contained in [G]. They are defined in terms of the expansion of the ambient or Poincaré metric associated to g in the sense of [FG1]. One searches for a smooth 1-parameter family of metrics h_r on M so that $h_0 = g$ and so that the metric

$$(1.1) \quad g_+ = \frac{dr^2 + h_r}{r^2}$$

on $M \times (0, \epsilon)$ is an asymptotic solution to $\text{Ric}(g_+) = -ng_+$ at $r = 0$. This together with the condition that h_r be even in r uniquely determines h_r to infinite order if n is odd, however only to order n if n is even, at which point there is a formal obstruction to finding a solution to the next order. The trace part of the Taylor coefficient at order n is determined but the determination of the trace-free part is obstructed by a trace-free symmetric 2-tensor called the ambient obstruction tensor.

Since h_r is even in r , it is natural to introduce a new variable $\rho = -\frac{1}{2}r^2$ and set $g_\rho = h_r$. The ambient metric coefficients are the determined Taylor coefficients $\partial_\rho^k g_\rho|_{\rho=0}$. These are given locally in terms of the initial metric $g_0 = g$; each of them can be written as a polynomial natural tensor expressible in terms of the curvature tensor of g and its covariant derivatives. The renormalized volume coefficients are defined by the expansion of the volume form:

$$(1.2) \quad \left(\frac{\det g_\rho}{\det g_0} \right)^{1/2} \sim 1 + \sum_{k=1}^{\infty} v_k \rho^k.$$

If n is odd, $v_k(g)$ is defined for all $k \geq 1$. If n is even, $v_k(g)$ is defined only for $k \leq n/2$ for general g , although $v_k(g)$ is defined for all $k \geq 1$ also in even dimensions if g is Einstein or locally conformally flat. A more detailed discussion is contained in §2.

The insight of Chang-Fang is to consider the $v_k(g)$ in the context of the properties satisfied by the $\sigma_k(g^{-1}P)$. Just comparing the formulae for these quantities shows that $v_k(g) = \sigma_k(g^{-1}P)$ if $k = 1$ or 2 . In [GJ] it is shown that this holds also for $k \geq 3$

if g is locally conformally flat. Moreover, $v_k(g)$ always satisfies the two properties discussed above which failed for $\sigma_k(g^{-1}P)$ for $3 \leq k \leq n$ for general metrics. One of the first important properties established of the v_k was that in dimension $n = 2k$, $\int_M v_k(g) dv_g$ is a conformal invariant for general metrics (a proof is given in [G]). And the new result of Chang-Fang is that the variational characterization holds for $v_k(g)$: for $n \neq 2k$, the equation $v_k(g) = c$ is the Euler-Lagrange equation for the functional $\int_M v_k(g) dv_g$ under conformal variations subject to the constraint $\text{Vol}_g(M) = 1$. This collection of facts suggests a strong parallel between the $v_k(g)$ and the $\sigma_k(g^{-1}P)$, and even that from some points of view the $v_k(g)$ have better properties.

However, study of the $v_k(g)$ involves significant challenges not shared by the $\sigma_k(g^{-1}P)$. Firstly, for $k \geq 3$, $v_k(g)$ depends on derivatives of the curvature of g . In fact, for $k \geq 2$, $v_k(g)$ depends on derivatives of curvature of order up to $2k - 4$. Secondly, the $v_k(g)$ are defined via an indirect, highly nonlinear, inductive algorithm: first one solves the Einstein equation formally to determine g_ρ and then expands its volume form to obtain $v_k(g)$. They are algebraically complicated and no explicit formula is known for general k .

A formula for v_3 was given in [GJ]; it is not difficult to carry out the algorithm explicitly by hand to this order. The result is:

$$(1.3) \quad v_3(g) = \sigma_3(g^{-1}P) + \frac{1}{3(n-4)} P^{ij} B_{ij},$$

where B_{ij} denotes the Bach tensor of g . It is well-known that under conformal change $\hat{g} = e^{2\omega}g$, the transformation law of the Bach tensor involves just first derivatives of the conformal factor. Thus an immediate consequence of (1.3) and the conformal transformation law

$$(1.4) \quad \hat{P}_{ij} = P_{ij} - \omega_{ij} + \omega_i \omega_j - \frac{1}{2} \omega_k \omega^k g_{ij}$$

of the Schouten tensor is the fact that the transformation law of v_3 involves at most second order derivatives of ω . Thus for a fixed metric g , the equation $v_3(e^{2\omega}g) = c$ is second order in ω . It is this equation that Chang-Fang propose to study by analogy with the σ_k -Yamabe problem.

In this paper, it is proved that the conformal transformation law involves at most second order derivatives of ω for all the v_k , as well as for all the ambient metric coefficients.

Theorem 1.1. *Under conformal change $\hat{g} = e^{2\omega}g$, the conformal transformation laws of the $\partial_\rho^k g_\rho|_{\rho=0}$ and the v_k involve at most second derivatives of ω . If n is odd, this is true for all k . If n is even, it is true for $\partial_\rho^k g_{ij}|_{\rho=0}$ for $1 \leq k \leq n/2 - 1$, and for $g^{ij} \partial_\rho^{n/2} g_{ij}|_{\rho=0}$ and v_k for $1 \leq k \leq n/2$.*

We give two different proofs of Theorem 1.1, each of which yields further information. The first proof proceeds by establishing that each of the determined

ambient metric coefficients $\partial_\rho^k g_{ij}|_{\rho=0}$ can be written in terms of simpler building blocks, each of which has a conformal transformation law involving at most second derivatives of ω . The building blocks consist of the Schouten tensor and a family $\Omega_{ij}^{(k)}$ of trace-free symmetric natural 2-tensors which we call the extended obstruction tensors. The $\Omega_{ij}^{(k)}$ are defined for all $k \geq 1$ if n is odd, but only for $1 \leq k \leq n/2 - 2$ if n is even. The name derives from the fact that when the dimension is viewed as a formal parameter, $\Omega_{ij}^{(k)}$ has a simple pole at dimension $n = 2k + 2$ whose residue is a multiple of the obstruction tensor in that dimension. For example,

$$(1.5) \quad \Omega_{ij}^{(1)} = \frac{1}{4-n} B_{ij},$$

and the obstruction tensor in dimension 4 is the Bach tensor B_{ij} . The result asserting that the ambient metric coefficients can be written in terms of the building blocks is the following.

Theorem 1.2. *Let $k \geq 1$. There is a linear combination $\mathcal{G}_{ij}^{(k)}(P, \Omega^{(1)}, \dots, \Omega^{(k-1)})$ of partial contractions with respect to g^{-1} of the Schouten tensor P and the $\Omega^{(l)}$, $1 \leq l \leq k-1$, such that the coefficients of $\mathcal{G}_{ij}^{(k)}$ are independent of n , and such that the ambient metric coefficients in dimension n are given by:*

$$(1.6) \quad \partial_\rho^k g_{ij}|_{\rho=0} = \mathcal{G}_{ij}^{(k)}(P, \Omega^{(1)}, \dots, \Omega^{(k-1)}),$$

for all $k \geq 1$ if n is odd and for $1 \leq k \leq n/2 - 1$ if n is even. Additionally, if $k \geq 2$, there is a linear combination $\mathcal{T}^{(k)}(P, \Omega^{(1)}, \dots, \Omega^{(k-2)})$ of complete contractions of the indicated tensors whose coefficients are independent of n , such that in even dimension n , one has

$$(1.7) \quad g^{ij} \partial_\rho^{n/2} g_{ij}|_{\rho=0} = \mathcal{T}^{(n/2)}(P, \Omega^{(1)}, \dots, \Omega^{(n/2-2)}).$$

A corollary is the analogous result for the renormalized volume coefficients.

Corollary 1.3. *Let $k \geq 1$. There is a linear combination $\mathcal{V}_k(P, \Omega^{(1)}, \dots, \Omega^{(k-2)})$ of complete contractions with respect to g^{-1} of the Schouten tensor P and the $\Omega^{(l)}$, $1 \leq l \leq k-2$, such that the coefficients of \mathcal{V}_k are independent of n , and such that the renormalized volume coefficients in dimension n are given by*

$$v_k(g) = \mathcal{V}_k(P, \Omega^{(1)}, \dots, \Omega^{(k-2)}),$$

for all $k \geq 1$ if n is odd and for $1 \leq k \leq n/2$ if n is even.

For example, (1.3) and (1.5) give

$$v_3(g) = \sigma_3(g^{-1}P) - \frac{1}{3} P^{ij} \Omega_{ij}^{(1)}.$$

The proof of Theorem 1.2 gives a fairly simple, direct algorithm for the inductive determination of the $\mathcal{G}_{ij}^{(k)}$ which is independent of the formal solution of the Einstein

equation. It is easy to carry this out to exhibit $\mathcal{G}_{ij}^{(k)}$ for small k ; we give the result for $k \leq 5$. The more significant algebraic complexity occurs in the expressions for the $\Omega_{ij}^{(k)}$ in terms of the curvature of g , for which solution of the Einstein equation is required and in which the dimension enters explicitly.

The extended obstruction tensors are part of the theory of conformal curvature tensors developed in §6 of [FG2]; they are particular instances of conformal curvature tensors. In particular, each of them has the property shared by all conformal curvature tensors that its conformal transformation law can be written explicitly in terms of other conformal curvature tensors and first derivatives of the conformal factor. Thus Theorem 1.1 follows immediately from Theorem 1.2 and Corollary 1.3. Moreover, this shows that the only way second derivative terms in ω can arise in the conformal transformation law of $v_k(g)$ is from occurrences in \mathcal{V}_k of the Schouten tensor.

A closer analysis of the conformal transformation law of the $\Omega_{ij}^{(k)}$ and of the form of the \mathcal{V}_k gives the following result describing the structure of $v_k(e^{2\omega}g)$ as a second order fully nonlinear operator.

Theorem 1.4. *Let $k \geq 1$ and suppose $k \leq n/2$ if n is even. Then*

$$e^{2k\omega} v_k(\widehat{g}) = \sigma_k(g^{-1}\widehat{P}) + \sum_{m=0}^{k-2} r_{k,m}(x, \nabla\omega, \widehat{P}),$$

where $r_{k,m}(x, \nabla\omega, \widehat{P})$ is a polynomial in $(\omega_i, \widehat{P}_{ij})$ which is homogeneous of degree m in \widehat{P} , of degree $\leq 2k - 2m - 2$ in $\nabla\omega$, and with coefficients depending on g .

Here \widehat{P} is the conformally transformed Schouten tensor given by (1.4).

Theorem 1.4 shows that $\sigma_k(g^{-1}\widehat{P})$ can be viewed as the leading term in $v_k(\widehat{g})$ from two points of view (at least for $k \leq n$ so that $\sigma_k \neq 0$). First, it has the highest homogeneity degree in \widehat{P} , which contains all the second derivative terms. Second, it contains all the terms with the highest total number of derivatives of ω . By this we mean that we expand $e^{2k\omega} v_k(\widehat{g})$ as a polynomial in ω_i, ω_{ij} and add up the total number derivatives on ω in each monomial. For example, in (1.4), each of the terms ω_{ij} , $\omega_i \omega_j$ and $\omega_k \omega^k$ has a total of 2 derivatives of ω . Thus $\sigma_k(g^{-1}\widehat{P})$ contains terms with $2k$ derivatives of ω . Theorem 1.4 implies that each term in each $r_{k,m}(x, \nabla\omega, \widehat{P})$ involves at most $2k - 2$ derivatives of ω . It is tempting to speculate that requiring these two properties gives a reasonable definition of a “principal part” of a second order fully nonlinear operator depending polynomially on the derivatives.

The second proof of Theorem 1.1 proceeds via a study of the linearization of $v_k(e^{2\omega}g)$ as a function of ω , i.e. of the linearized conformal transformation law of $v_k(g)$. The main ingredient is a formula for the conformal variation of the 1-parameter family h_r of metrics on M which arise when a given asymptotically

hyperbolic metric g_+ is written in the form (1.1). If one chooses a conformally related boundary metric $\widehat{g} = e^{2\omega}g$, then up to a diffeomorphism of $M \times [0, \epsilon)$, g_+ can be written in the form (1.1) with a different 1-parameter family \widehat{h}_r satisfying $\widehat{h}_0 = \widehat{g}$. It is possible to solve explicitly for the infinitesimal diffeomorphism in terms of ω and then for the infinitesimal conformal variation of h_r . Letting δ denote infinitesimal conformal variation, the result when written in terms of $g_\rho = h_r$ defined as above is the following: (it is convenient to use also the notation $g_{ij}(\rho)$ or simply g_{ij} for g_ρ)

$$(1.8) \quad (\delta g)_{ij} = 2\omega(1 - \rho \partial_\rho)g_{ij} + 2\nabla_{(i}Y_{j)}$$

with

$$(1.9) \quad Y^i(\rho) = - \int_0^\rho g^{ij}(u) du \partial_j \omega, \quad Y_j(\rho) = g_{ij}(\rho)Y^i(\rho).$$

Here ∇_i denotes the covariant derivative with respect to g_ρ with ρ fixed. An easy consequence of this is a formula for the infinitesimal conformal variation of the $v_k(g)$. Set

$$(1.10) \quad v(\rho) = \left(\frac{\det g_\rho}{\det g_0} \right)^{1/2}.$$

Theorem 1.5. *Let $k \geq 1$ and $k \leq n/2$ if n is even. The infinitesimal conformal variation of v_k is given by:*

$$(1.11) \quad \delta v_k = -2k\omega v_k + \nabla_i \left(L_{(k)}^{ij} \nabla_j \omega \right),$$

where

$$(1.12) \quad L_{(k)}^{ij} = -\frac{1}{k!} \partial_\rho^k \left(v(\rho) \int_0^\rho g^{ij}(u) du \right) \Big|_{\rho=0} = - \sum_{l=1}^k \frac{1}{l!} v_{k-l} \partial_\rho^{l-1} g^{ij} \Big|_{\rho=0}.$$

In (1.11), ∇_i denotes the covariant derivative with respect to the initial metric $g = g_0$.

The infinitesimal transformation laws (1.8) and (1.11) clearly involve derivatives of ω of order at most 2. The second proof of Theorem 1.1 proceeds by arguing that if the infinitesimal conformal transformation law of a natural tensor involves at most m derivatives of ω for some $m \geq 0$, then the same is true of the full transformation law. This is the content of Proposition 3.6.

The Chang-Fang variational characterization of the equations $v_k(g) = c$ as Euler-Lagrange equations if $n \neq 2k$ is an easy consequence of (1.11). The main point is that the second term on the right hand side of (1.11) is a divergence, which integrates to zero. Thus the only contribution to the Euler-Lagrange equation is the scaling contribution given by the first term, so that the Euler-Lagrange equation is $v_k(g) = c$. The proof of Chang-Fang is also based on using the diffeomorphism

invariance of g_+ under conformal change of g to generate a divergence term; these amount to different versions of the same proof. But by working directly with metrics in the normal form (1.1), it is possible to give explicit formulae for the divergence terms, among other things making it clear that these terms depend on no more than second derivatives of ω . The approach to the linearization formulae used here is the same as in [ISTY], where the formulae (1.8) and (1.11) already appear.

Formula (1.11) can be interpreted as identifying the linearization at $\omega = 0$ of the second order fully nonlinear operator $v_k(e^{2\omega}g)$ with g fixed. In particular, the linearization is exhibited in divergence form modulo the zeroth order scaling term, and its principal part is $L_{(k)}^{ij} \nabla_{ij}^2 \omega$. This may be useful in determining ellipticity of $v_k(e^{2\omega}g) = c$. However, although all of these linearization formulae are explicit, they are written in terms of the coefficients in the ambient metric and renormalized volume expansions, and therefore are difficult to understand directly in terms of geometry of g . Additionally, they involve more and more derivatives of g as k increases.

The results obtained in this paper support and extend the suggestion of Chang-Fang that the $v_k(g)$ are worthy of further study. However, significant algebraic complications remain and the geometric content of the equations $v_k(e^{2\omega}g) = c$ is unclear, particularly for large k . Perhaps it would be reasonable to try to extend directly the analytic theory of the σ_k equations to elliptic fully nonlinear equations allowing “lower order terms” with structure as in Theorem 1.4.

2. EXTENDED OBSTRUCTION TENSORS

We begin this section by recalling the Poincaré metric expansion and the definition of the renormalized volume coefficients. These are then reformulated in terms of the expansion of the ambient metric. After reviewing the theory of conformal curvature tensors from [FG2], we define the extended obstruction tensors as certain specific conformal curvature tensors. We establish the basic properties of the extended obstruction tensors. Then we prove Theorems 1.2 and 1.4. The section is concluded by giving some explicit formulae for small k .

First recall the Poincaré metric expansion and the definition of the renormalized volume coefficients v_k . References for this material are [G], [GH], and [FG2]. Let g be a metric of signature (p, q) on a manifold M of dimension $n \geq 3$. There are versions of most of the statements in dimension 2, but this case is anomalous and our main interest is in higher dimensions, so for simplicity we assume $n \geq 3$. If n is odd, there is a smooth 1-parameter family h_r , $0 \leq r < 1$, of metrics on M such that $h_0 = g$ and such that the metric $g_+ = r^{-2}(dr^2 + h_r)$ of signature $(p+1, q)$ on $M \times (0, 1)$ satisfies that $\text{Ric}(g_+) + ng_+$ vanishes to infinite order at $r = 0$. The Taylor expansion in r of h_r at $r = 0$ can be chosen to be even in r , in which case it is uniquely determined.

For $n \geq 4$ even, the corresponding statement holds only to a finite order. We say that a tensor is $O(r^m)$ if all of its components relative to a frame smooth up to $r = 0$ are $O(r^m)$. We use lower case Latin indices to label objects on M . When n is even, h_r can be chosen so that its Taylor expansion is even in r and such that

$$\text{Ric}(g_+) + ng_+ = O(r^{n-2}), \quad h^{ij} (\text{Ric}(g_+) + ng_+)_{ij} = O(r^n).$$

In the second equation, $(\text{Ric}(g_+) + ng_+)_{ij}$ denotes the component with both indices in the M factor. In calculating the trace, h^{ij} can be taken to be either $(h_0)^{ij} = g^{ij}$ or $(h_r)^{ij}$. These conditions uniquely determine $h_r \bmod O(r^n)$ and also $\text{tr}_g h_r \bmod O(r^{n+2})$. For $n \geq 4$ even, there is a conformally invariant trace-free, divergence-free natural tensor \mathcal{O}_{ij} , the ambient obstruction tensor, which obstructs the existence of a formal power series solution for g_+ to the next order. \mathcal{O}_{ij} depends on derivatives of g of order up to n . When $n = 4$, \mathcal{O}_{ij} is the classical Bach tensor.

The passage from g to g_+ is conformally invariant in the sense that if $\widehat{g} = e^{2\omega}g$ with $\omega \in C^\infty(M)$, and \widehat{h}_r denotes the expansion determined by \widehat{g} , then the metrics g_+ and $\widehat{g}_+ = r^{-2} \left(dr^2 + \widehat{h}_r \right)$ are isometric by a diffeomorphism restricting to the identity on $M \times \{0\}$, to infinite order if n is odd, and up to a term which is $O(r^{n-2})$ and the trace of whose tangential component is $O(r^n)$ if n is even.

There are two special families of conformal structures in even dimensions for which the obstruction tensor vanishes and for which it is possible to uniquely determine the expansion of the Poincaré metric to infinite order in a conformally invariant way. These are the locally conformally flat structures and the conformal classes containing an Einstein metric. In these cases, the normalized expansion can be written explicitly and terminates at order four: for all $n \geq 3$, one has

$$(2.1) \quad (h_r)_{ij} = g_{ij} - P_{ij}r^2 + \frac{1}{4}P_{ik}P_j^k r^4$$

if g is Einstein or locally conformally flat. See [SS] and §7 of [FG2].

The volume form of g_+ is

$$dv_{g_+} = r^{-n-1} dv_{h_r} dr = r^{-n-1} \left(\frac{\det h_r}{\det h_0} \right)^{1/2} dv_g dr.$$

The renormalized volume coefficients are defined by the Taylor expansion:

$$(2.2) \quad \left(\frac{\det h_r}{\det h_0} \right)^{1/2} \sim 1 + \sum_{k=1}^{\infty} (-2)^{-k} v_k r^{2k}.$$

Thus v_k is uniquely determined by $g = h_0$ for all $k \geq 1$ if n is odd, and for $1 \leq k \leq n/2$ if n is even. As will be discussed in more detail below, v_1 and v_2 are given by

$$v_1 = J, \quad v_2 = \frac{1}{2} (J^2 - P_{ij}P^{ij}),$$

where $J = R/2(n-1) = P_i^i$. If g is Einstein or locally conformally flat, then the v_k are determined by g for all k for both n even and odd. Proposition 1 of [GJ] uses (2.1) to show that $v_k(g) = \sigma_k(g^{-1}P)$ for all $k \geq 1$ if g is locally conformally flat. The same argument shows that this also holds if g is Einstein. In particular, $v_k(g)$ is constant for Einstein metrics.

The evenness of the Poincaré metric in r suggests to introduce r^2 as a new variable. Set $\rho = -\frac{1}{2}r^2$ and $g_\rho = h_r$. Then the volume expansion (2.2) becomes (1.2).

The 1-parameter family g_ρ can be characterized directly in terms of the expansion of the ambient metric \tilde{g} associated to g , which is equivalent to the expansion of the Poincaré metric g_+ . Define \tilde{g} , a metric of signature $(p+1, q+1)$ on $\mathbb{R}_+ \times M \times (-1/2, 0] \ni (t, x, \rho)$, by

$$(2.3) \quad \tilde{g} = 2t \, dt \, d\rho + 2\rho \, dt^2 + t^2 g_\rho.$$

The condition $\text{Ric}(g_+) + n g_+ = 0$ is equivalent to $\text{Ric}(\tilde{g}) = 0$. Thus the expansion of g_ρ can be thought of as arising from formally solving $\text{Ric}(\tilde{g}) = 0$ to the appropriate order rather than $\text{Ric}(g_+) + n g_+ = 0$. An advantage of considering \tilde{g} is that \tilde{g} is smooth near $\rho = 0$, whereas g_+ is singular at $r = 0$. As we will see, this makes it easier to pass objects constructed out of \tilde{g} back to M .

For each $k \geq 1$ satisfying also $k < n/2$ if n is even, the Taylor coefficient $\partial_\rho^k g_\rho|_{\rho=0}$ is given by a polynomial natural tensor depending on the initial metric g . For n even, the trace $g^{ij} \left(\partial_\rho^{n/2} g_{ij} \right)|_{\rho=0}$ at order $n/2$ is also a natural scalar invariant of g . It is possible to directly compute the beginning coefficients. For example, letting $' = \partial_\rho$ and suppressing the ρ in g_ρ , (3.6) and (3.18) of [FG2] show that one has at $\rho = 0$:

$$(2.4) \quad g'_{ij} = 2P_{ij}, \quad g''_{ij} = \frac{2}{4-n} B_{ij} + 2P_i^k P_{kj},$$

where

$$B_{ij} = P_{ij,k}^k - P_{ik,j}^k - P^{kl} W_{kijl}$$

is the Bach tensor. Here W_{ijkl} denotes the Weyl tensor.

The last part of §6 of [FG2] considers a family of trace-free symmetric natural 2-tensors depending on a metric g . Here we call these the extended obstruction tensors. They have the feature that their transformation laws under conformal change are explicit and relatively simple. The first extended obstruction tensor is $(4-n)^{-1} B_{ij}$, which we denote by $\Omega_{ij}^{(1)}$. Its well-known transformation law under conformal change $\hat{g} = e^{2\omega} g$ is:

$$(2.5) \quad e^{2\omega} \hat{\Omega}_{ij}^{(1)} = \Omega_{ij}^{(1)} - 2\omega^k C_{(ij)k} + \omega^k \omega^l W_{kijl},$$

where $C_{ijk} = P_{ij,k} - P_{ik,j}$ is the Cotton tensor. Equation (2.4) shows that $g''_{ij}|_{\rho=0}$ can be expressed in terms of $\Omega_{ij}^{(1)}$ by:

$$(2.6) \quad \frac{1}{2}g''_{ij}|_{\rho=0} = \Omega_{ij}^{(1)} + P_i^k P_{kj}.$$

The definition and basic properties of the extended obstruction tensors are part of the theory of conformal curvature tensors developed in §6 of [FG2]. We summarize the relevant considerations and refer to [FG2] for details.

Consider the curvature tensor and its covariant derivatives for an ambient metric (2.3). We denote its curvature tensor by \tilde{R} , with components \tilde{R}_{IJKL} . Here capital Latin indices are used for objects on $\mathbb{R}_+ \times M \times \mathbb{R} \ni (t, x, \rho)$, and we use '0' for the \mathbb{R}_+ factor (t component), lower case Latin indices for the M factor, and '∞' for the \mathbb{R} factor (ρ component). For $r \geq 0$, the r -th covariant derivative $\tilde{\nabla}^r \tilde{R}$ of the curvature tensor of \tilde{g} will be denoted $\tilde{R}^{(r)}$, with components $\tilde{R}_{IJKL, M_1 \dots M_r}^{(r)}$. Sometimes the superscript $^{(r)}$ is omitted when the list of indices makes clear the value of r .

The conformal curvature tensors are tensors on M obtained from the covariant derivatives of curvature of \tilde{g} as follows. Choose an order $r \geq 0$ of covariant differentiation. Divide the set of symbols $IJKLM_1 \dots M_r$ into three disjoint subsets labeled \mathcal{S}_0 , \mathcal{S}_M and \mathcal{S}_∞ . Set the indices in \mathcal{S}_0 equal to 0, those in \mathcal{S}_∞ equal to ∞ , and let those in \mathcal{S}_M correspond to M in the decomposition $\mathbb{R}_+ \times M \times \mathbb{R}$. Evaluate the resulting component $\tilde{R}_{IJKL, M_1 \dots M_r}$ at $\rho = 0$ and $t = 1$. This defines a tensor on M , sometimes denoted by $\tilde{R}_{\mathcal{S}_0, \mathcal{S}_M, \mathcal{S}_\infty}^{(r)}$, whose rank is the cardinality of the set \mathcal{S}_M . In local coordinates, the indices in \mathcal{S}_M vary between 1 and n .

The simplest case is $r = 0$. The curvature tensor \tilde{R} of \tilde{g} satisfies $\tilde{R}_{IJK0} = 0$, so we must choose $\mathcal{S}_0 = \emptyset$ in order to get a nonzero component. Up to reordering the indices, there are only three possible nonzero choices (see (6.2) of [FG2]):

$$(2.7) \quad \tilde{R}_{ijkl}|_{\rho=0, t=1} = W_{ijkl}, \quad \tilde{R}_{\infty jkl}|_{\rho=0, t=1} = C_{jkl} \quad \tilde{R}_{\infty ij\infty}|_{\rho=0, t=1} = \frac{B_{ij}}{4-n}.$$

Thus the conformal curvature tensors which arise for $r = 0$ are precisely the Weyl, Cotton, and Bach tensors of g , except that when $n = 4$, the Bach tensor arises as the obstruction tensor rather than as a conformal curvature tensor.

Since g_ρ is uniquely determined by $g_0 = g$ to infinite order for n odd, it follows that for n odd the conformal curvature tensors $\tilde{R}_{\mathcal{S}_0, \mathcal{S}_M, \mathcal{S}_\infty}^{(r)}$ for all choices of r and $\mathcal{S}_0, \mathcal{S}_M, \mathcal{S}_\infty$ are defined and are polynomial natural tensors. However, when n is even, one must restrict the orders of differentiation to avoid the indeterminacy of g_ρ at order $n/2$. For n even, the tensor $\tilde{R}_{\mathcal{S}_0, \mathcal{S}_M, \mathcal{S}_\infty}^{(r)}$ depends only on g and is a natural tensor so long as $s_M + 2s_\infty \leq n + 1$, where s_M, s_∞ are the cardinalities of $\mathcal{S}_M, \mathcal{S}_\infty$, resp. If g is Einstein or locally conformally flat, then also for n even the

conformal curvature tensors are defined for all choices of r and $\mathcal{S}_0, \mathcal{S}_M, \mathcal{S}_\infty$. As will be seen below, they all vanish in the locally conformally flat case.

Because \tilde{g} changes by a diffeomorphism when g is changed conformally, the covariant derivatives $\tilde{\nabla}^r \tilde{R}$ of ambient curvature transform tensorially under conformal change of g . This leads to an explicit identification of the transformation laws of the conformal curvature tensors under conformal change. The following is Proposition 6.5 of [FG2].

Proposition 2.1. *Let g and $\hat{g} = e^{2\omega} g$ be conformally related metrics on M . Let $IJKLM_1 \cdots M_r$ be a list of indices, s_0 of which are 0, s_M of which correspond to M , and s_∞ of which are ∞ . If n is even, assume that $s_M + 2s_\infty \leq n + 1$. Then the conformal curvature tensors satisfy the conformal transformation law:*

$$(2.8) \quad e^{2(s_\infty-1)\omega} \tilde{R}_{IJKL, M_1 \cdots M_r} |_{\hat{\rho}=0, \hat{t}=1} = \tilde{R}_{ABCD, F_1 \cdots F_r} |_{\rho=0, t=1} p^A_I \cdots p^{F_r}_{M_r},$$

where p^A_I is the matrix

$$(2.9) \quad p^A_I = \begin{pmatrix} 1 & \omega_i & -\frac{1}{2}\omega_k \omega^k \\ 0 & \delta^a_i & -\omega^a \\ 0 & 0 & 1 \end{pmatrix}.$$

Here the conformal curvature tensor $\tilde{R}_{IJKL, M_1 \cdots M_r} |_{\rho=0, t=1}$ evaluated for the metric \hat{g} is denoted $\hat{\tilde{R}}_{IJKL, M_1 \cdots M_r} |_{\hat{\rho}=0, \hat{t}=1}$. The variables $\hat{\rho}$ and \hat{t} denote the coordinates on $\mathbb{R}_+ \times M \times \mathbb{R}$, thought of as a separate copy from the space for the unhatted metric. In (2.9), indices on ω_i are raised using the initial metric g .

In expanding the right hand side of (2.8), the leading term arises by replacing each p by δ , giving $\tilde{R}_{IJKL, M_1 \cdots M_r} |_{\rho=0, t=1}$. Because of the upper-triangular form of the matrix p^A_I , the other terms on the right hand side all involve “earlier” conformal curvature tensors in the sense that each ‘ i ’ can be replaced only by 0 and each ∞ only by an ‘ i ’ or a 0. It is clear that the conformal transformation law of a conformal curvature tensor involves only other conformal curvature tensors and first derivatives of ω . In case $r = 0$, using (2.7) and the fact that $\tilde{R}_{IJK0} = 0$, one sees that (2.8) reproduces the conformal invariance of the Weyl tensor and the usual conformal transformation laws of the Cotton and Bach tensors. Equation (2.8) can be interpreted as asserting that $\tilde{\nabla}^r \tilde{R} |_{\rho=0}$ defines a section of the $(r+4)$ -th tensor power of the cotractor bundle of the conformal manifold $(M, [g])$ with a particular conformal weight.

It follows directly from the definition that the conformal curvature tensors all vanish if g is flat. Thus a consequence of (2.8) is that also they all vanish if g is locally conformally flat. By the infinite order invariance of the ambient metric for n even in the locally conformally flat case, this is true for all conformal curvature tensors in both even and odd dimensions.

We now define the extended obstruction tensors.

Definition 2.2. Let $k \geq 1$. Suppose that n is odd or n is even and $n > 2(k+1)$. Define the k -th extended obstruction tensor $\Omega_{ij}^{(k)}$ to be the conformal curvature tensor:

$$\Omega_{ij}^{(k)} = \widetilde{R}_{\infty ij \infty, \underbrace{\infty \dots \infty}_{k-1}}|_{\rho=0, t=1}.$$

According to the above discussion, $\Omega_{ij}^{(k)}$ is a polynomial natural tensor of the initial metric g . For $k=1$, (2.7) shows that $\Omega_{ij}^{(1)}$ is given by (1.5). It is clear that $\Omega_{ij}^{(k)}$ is symmetric in ij , and it is also trace-free:

Proposition 2.3. *For each $k \geq 1$ and in all dimensions n as above for which $\Omega_{ij}^{(k)}$ is defined, one has*

$$g^{ij}\Omega_{ij}^{(k)} = 0.$$

Proof. This is a consequence of the Ricci-flatness of the ambient metric (to the appropriate order if n is even). First suppose that n is odd. Since $\text{Ric}(\widetilde{g}) = O(\rho^\infty)$, we have

$$(2.10) \quad \widetilde{g}^{IJ} \widetilde{R}_{KIJL, M_1 \dots M_r} = 0$$

at $\rho = 0$ for all choices of $KLM_1 \dots M_r$. Take all of KLM_1, \dots, M_r to be ∞ . At $\rho = 0$ we have

$$\widetilde{g}^{IJ} = \begin{pmatrix} 0 & 0 & t^{-1} \\ 0 & t^{-2}g^{ij} & 0 \\ t^{-1} & 0 & 0 \end{pmatrix}.$$

The terms in (2.10) with $IJ = 0\infty$ or $\infty 0$ vanish by skew-symmetry of \widetilde{R}_{KIJL} in KI and JL . Thus (2.10) reduces to $g^{ij}\widetilde{R}_{\infty ij \infty, \infty \dots \infty} = 0$ as desired.

The same argument applies if n is even, so long as one checks that the order vanishing of $\text{Ric}(\widetilde{g})$ is sufficient to under the restriction $n > 2(k+1)$. This is precisely the statement of Proposition 6.4 of [FG2]. \square

Since for g locally conformally flat, all conformal curvature tensors are defined and vanish whether n is even or odd, in particular it follows that $\Omega_{ij}^{(k)}$ is defined and $\Omega_{ij}^{(k)} = 0$ for all k for locally conformally flat g . This is also true if g is Einstein; see Proposition 7.6 of [FG2]. Note that general conformal curvature tensors do not vanish for Einstein metrics; for example the Weyl tensor is a conformal curvature tensor. For n even and g Einstein, the vanishing of $\Omega_{ij}^{(n/2-1)}$ is actually the condition used to normalize the indeterminacy in the ambient metric; see Proposition 7.7 of [FG2].

Each obstruction tensor has divergence zero. But this property does not extend to the extended obstruction tensors. Already this fails for $k=1$: the divergence of the Bach tensor is given by $B_{ij, j} = (n-4)P^{jk}C_{jki}$.

Next we define “higher Cotton tensors”, which will enter into the conformal transformation law of the extended obstruction tensors.

Definition 2.4. Let $k \geq 1$. Suppose that n is odd or n is even and $n \geq 2(k+1)$. Define the k -th higher Cotton tensor $C_{ijl}^{(k)}$ by:

$$C_{ijl}^{(k)} = 2\tilde{R}_{\infty(ij)l, \underbrace{\infty \dots \infty}_{k-1}} + \tilde{R}_{\infty ij \infty, \underbrace{l \infty \dots \infty}_{k-1}} + \tilde{R}_{\infty ij \infty, \underbrace{\infty l \infty \dots \infty}_{k-1}} + \dots + \tilde{R}_{\infty ij \infty, \underbrace{\infty \dots \infty l}_{k-1}}.$$

Here all \tilde{R} components are evaluated at $\rho = 0$, $t = 1$.

For k and n as in Definition 2.4, $C_{ijl}^{(k)}$ is a polynomial natural tensor of the initial metric g . As for the extended obstruction tensors, it is defined and vanishes for all k for g Einstein or locally conformally flat. Equation (2.7) shows that

$$C_{ijl}^{(1)} = 2C_{(ij)l}.$$

The tensors C_{ijl} and $C_{ijl}^{(1)}$ are equivalent; C_{ijl} can be recovered from $C_{ijl}^{(1)}$ by $C_{ijl} = \frac{2}{3}C_{ijl}^{(1)}$. It is clear that $C_{ijl}^{(k)}$ is symmetric in ij , and it is also trace-free in these indices:

Proposition 2.5. *For each $k \geq 1$ and in all dimensions n as above for which $C_{ijl}^{(k)}$ is defined, one has*

$$g^{ij}C_{ijl}^{(k)} = 0.$$

Proof. The proof is similar to that of Proposition 2.3. Again assume first that n is odd. Take K , L , and all but one of the M_s to be ∞ in (2.10) to deduce just as in the proof of Proposition 2.3 that $g^{ij}\tilde{R}_{\infty ij \infty, \underbrace{\infty \dots l \dots \infty}_{k-1}} = 0$ for any location of the

index l after the comma. The same argument applied to the first term on the right hand side in Definition 2.4 shows that at $\rho = 0$ and $t = 1$ we have

$$g^{ij}\tilde{R}_{\infty ij l, \underbrace{\infty \dots \infty}_{k-1}} + \tilde{R}_{\infty 0 \infty l, \underbrace{\infty \dots \infty}_{k-1}} = 0.$$

Now (1) of Proposition 6.1 of [FG2] states that

$$\tilde{R}_{IJK0, M_1 \dots M_r} = - \sum_{s=1}^r \tilde{R}_{IJK M_s, M_1 \dots \widehat{M_s} \dots M_r}$$

at $t = 1$. Applying this along with the symmetries of \tilde{R} shows that $\tilde{R}_{\infty 0 \infty l, \underbrace{\infty \dots \infty}_{k-1}} =$

$\tilde{R}_{\infty l \infty 0, \underbrace{\infty \dots \infty}_{k-1}} = 0$, and the result follows.

Proposition 6.4 of [FG2] shows that the same argument applies if n is even and $n \geq 2(k+1)$. \square

We remark that $g^{jl}C_{ijl}^{(k)} = 0$ for $1 \leq k \leq 3$, but not for $k = 4$. Also, the symmetry $C_{(ijl)}^{(1)} = 0$ satisfied by $C_{ijl}^{(1)} = 2C_{(ij)l}$ does not hold for $C_{ijl}^{(2)}$.

A special case of Proposition 2.1 is the conformal transformation law for the extended obstruction tensors:

Proposition 2.6. *Let $k \geq 1$. Let n be odd or even with $n > 2(k+1)$. Under a conformal change $\widehat{g} = e^{2\omega}g$, the conformally transformed extended obstruction tensor is given by:*

$$e^{2k\omega}\widehat{\Omega}_{ij}^{(k)} = \Omega_{ij}^{(k)} + \sum' \widetilde{R}_{ABCD, F_1 \dots F_{k-1}}|_{\rho=0, t=1} p^A_{\infty} p^B_i p^C_j p^D_{\infty} p^{F_1}_{\infty} \dots p^{F_{k-1}}_{\infty},$$

where p^A_I is given by (2.9) and \sum' denotes the sum over all indices except for $ABCD F_1 \dots F_{k-1} = \infty i j \infty \infty \dots \infty$.

Thus the conformal transformation law of the extended obstruction tensors is given explicitly in terms of conformal curvature tensors and first derivatives of the conformal factor. For $k = 1$, this reproduces (2.5). By the upper-triangular form of p^A_I , all of the conformal curvature tensors appearing in \sum' with nonzero coefficient are defined if n is even and $n \geq 2(k+1)$. Next we identify the terms in the transformation law which are linear in $\nabla\omega$.

Proposition 2.7. *Let k, n be as in Proposition 2.6. Under conformal change $\widehat{g} = e^{2\omega}g$, we have:*

$$e^{2k\omega}\widehat{\Omega}_{ij}^{(k)} = \Omega_{ij}^{(k)} - \omega^l C_{ijl}^{(k)} + O(|\nabla\omega|^2).$$

Proof. For a term in \sum' in Proposition 2.6 to be linear in $\nabla\omega$, all p 's but one must be δ , and p^0_{∞} terms are excluded. If we suppress writing $|_{\rho=0, t=1}$, we obtain:

$$\begin{aligned} & e^{2k\omega}\widehat{\Omega}_{ij}^{(k)} - \Omega_{ij}^{(k)} \\ &= -\omega^l \left(\widetilde{R}_{lij\infty, \underbrace{\infty \dots \infty}_{k-1}} + \widetilde{R}_{\infty ijl, \underbrace{\infty \dots \infty}_{k-1}} + \widetilde{R}_{\infty ij\infty, \underbrace{l\infty \dots \infty}_{k-1}} + \dots + \widetilde{R}_{\infty ij\infty, \underbrace{\infty \dots \infty l}_{k-1}} \right) \\ & \quad + \omega_i \widetilde{R}_{\infty 0j\infty, \underbrace{\infty \dots \infty}_{k-1}} + \omega_j \widetilde{R}_{\infty i0\infty, \underbrace{\infty \dots \infty}_{k-1}} + O(|\nabla\omega|^2) \\ &= -\omega^l C_{ijl}^{(k)} + \omega_i \widetilde{R}_{\infty 0j\infty, \underbrace{\infty \dots \infty}_{k-1}} + \omega_j \widetilde{R}_{\infty i0\infty, \underbrace{\infty \dots \infty}_{k-1}} + O(|\nabla\omega|^2). \end{aligned}$$

However, $\widetilde{R}_{\infty 0j\infty, \underbrace{\infty \dots \infty}_{k-1}} = \widetilde{R}_{\infty i0\infty, \underbrace{\infty \dots \infty}_{k-1}} = 0$ as in the proof of Proposition 2.5, and the result follows. \square

It is possible to view the dimension as a formal parameter and thus regard each of the extended obstruction tensors as a natural tensor depending rationally on n ; see the discussion at the end of §6 of [FG2] (where, however, n is called d). The

following result, which is Proposition 6.7 of [FG2], justifies the name “extended obstruction tensor”.

Proposition 2.8. *Viewed as a natural tensor rational in the dimension n , $\Omega_{ij}^{(k)}$ has a simple pole at $n = 2(k + 1)$ with residue given by*

$$\text{Res}_{n=2(k+1)} \Omega_{ij}^{(k)} = (-1)^k [2^{k-1}(k-1)!]^{-1} \mathcal{O}_{ij},$$

where \mathcal{O}_{ij} denotes the obstruction tensor in dimension $2(k + 1)$.

As noted above, in the transformation law in Proposition 2.6, all of the conformal curvature tensors appearing in \sum' with nonzero coefficient are regular at $n = 2(k + 1)$. Therefore, formally taking the residue of this transformation law at $n = 2(k + 1)$ recovers the conformal invariance of the obstruction tensor in dimension $2(k + 1)$. Likewise, for $k > 1$ we may consider the behavior as $n \rightarrow 2l$ with $2 \leq l \leq k$. It can be shown that $\Omega_{ij}^{(k)}$ and all the conformal curvature tensors appearing in its transformation law have at most simple poles at $n = 2k$. It is possible to justify the relation obtained by formally taking the residue at $n = 2k$ in the transformation law for $\Omega_{ij}^{(k)}$; this gives the conformal transformation law of $\text{Res}_{n=2k} \Omega_{ij}^{(k)}$. In general, the order of the poles increases with $k - l$. For example, $\Omega_{ij}^{(3)}$ has a double pole at $n = 4$, with leading coefficient a nonzero multiple of $B_i^k B_{kj}$. In this case, consideration of the coefficient of $(n - 4)^{-2}$ in the transformation law in Proposition 2.6 recovers the conformal invariance of $B_i^k B_{kj}$ in dimension 4.

Now we turn to the proof of Theorem 1.2, which asserts that the Taylor coefficients in the ambient metric expansion can be written in terms of the Schouten tensor and the extended obstruction tensors by formulae universal in the dimension.

Proof of Theorem 1.2. We prove by induction on k a stronger statement holding not only at $\rho = 0$. Consider a metric \tilde{g} of the form (2.3), where now g_ρ is any smooth 1-parameter family of metrics on M , i.e. we make no assumption that \tilde{g} is asymptotically Ricci-flat. For $k \geq 1$, define

$$\Lambda_{ij}^{(k)} = \widetilde{R}_{\infty ij \infty, \underbrace{\infty \dots \infty}_{k-1}}|_{t=1},$$

a family of symmetric 2-tensors on M parametrized by ρ . We claim that for each $k \geq 1$, there is a linear combination $\mathcal{Q}_{ij}^{(k)}$ of partial contractions with respect to g_ρ^{-1} of g'_ρ and the $\Lambda^{(l)}$, $1 \leq l \leq k - 1$, whose coefficients are independent of n , such that the identity

$$(2.11) \quad \partial_\rho^k g_{ij} = \mathcal{Q}_{ij}^{(k)}(g', \Lambda^{(1)}, \dots, \Lambda^{(k-1)})$$

holds for all ρ . Since for \tilde{g} asymptotically Ricci-flat, we have $g'|_{\rho=0} = 2P$ and $\Lambda^{(l)}|_{\rho=0} = \Omega^{(l)}$ (for $l < n/2 - 1$ if n is even), the first statement of Theorem 1.2 follows upon setting $\rho = 0$.

Case $k = 1$ of (2.11) is trivial taking $\mathcal{Q}_{ij}^{(1)} = g'_{ij}$. For $k = 2$, we use an explicit calculation of the component $\tilde{R}_{\infty ij\infty}$ of a metric (2.3). The Christoffel symbols of \tilde{g} can be written explicitly; see (3.16) of [FG2]. From this it is straightforward to calculate the curvature tensor of \tilde{g} ; see (6.1) of [FG2]. One obtains in particular

$$(2.12) \quad \tilde{R}_{\infty ij\infty}|_{t=1} = \frac{1}{2} \left(g''_{ij} - \frac{1}{2} g^{kl} g'_{ik} g'_{jl} \right).$$

Thus

$$(2.13) \quad g''_{ij} = 2\Lambda_{ij}^{(1)} + \frac{1}{2} g^{kl} g'_{ik} g'_{jl},$$

which is a relation of the form (2.11) for $k = 2$.

We need a preliminary calculation before proceeding with the induction argument. The calculation of the covariant derivative in terms of Christoffel symbols gives

$$\begin{aligned} \tilde{R}_{\infty ij\infty, \underbrace{\infty \dots \infty}_{k+1}} &= \partial_\rho \tilde{R}_{\infty ij\infty, \underbrace{\infty \dots \infty}_k} - \tilde{\Gamma}_{\infty\infty}^A \tilde{R}_{Aij\infty, \underbrace{\infty \dots \infty}_k} - \tilde{\Gamma}_{i\infty}^A \tilde{R}_{\infty A j\infty, \underbrace{\infty \dots \infty}_k} \\ &\quad - \tilde{\Gamma}_{j\infty}^A \tilde{R}_{\infty i A\infty, \underbrace{\infty \dots \infty}_k} - \tilde{\Gamma}_{\infty\infty}^A \tilde{R}_{\infty i j A, \underbrace{\infty \dots \infty}_k} \\ &\quad - \tilde{\Gamma}_{\infty\infty}^A \tilde{R}_{\infty i j\infty, \underbrace{A \dots \infty}_k} - \dots - \tilde{\Gamma}_{\infty\infty}^A \tilde{R}_{\infty i j\infty, \underbrace{\infty \dots A}_k}. \end{aligned}$$

Now (3.16) of [FG2] shows that these Christoffel symbols are given by:

$$\tilde{\Gamma}_{\infty\infty}^A = 0 \quad \text{for all } A$$

and

$$\tilde{\Gamma}_{i\infty}^0 = 0, \quad \tilde{\Gamma}_{i\infty}^l = \frac{1}{2} g^{lm} g'_{im}, \quad \tilde{\Gamma}_{i\infty}^\infty = 0.$$

Therefore

$$(2.14) \quad \tilde{R}_{\infty ij\infty, \underbrace{\infty \dots \infty}_{k+1}} = \partial_\rho \tilde{R}_{\infty ij\infty, \underbrace{\infty \dots \infty}_k} - \frac{1}{2} g^{lm} g'_{im} \tilde{R}_{\infty l j\infty, \underbrace{\infty \dots \infty}_k} - \frac{1}{2} g^{lm} g'_{jm} \tilde{R}_{\infty i l\infty, \underbrace{\infty \dots \infty}_k}.$$

The ρ derivative commutes with restriction to $t = 1$, so this can be written in terms of the $\Lambda_{ij}^{(k)}$ as

$$(2.15) \quad \partial_\rho \Lambda_{ij}^{(k)} = \Lambda_{ij}^{(k+1)} + g^{lm} g'_{m(i} \Lambda_{j)l}^{(k)}.$$

Now we prove that there is an identity of the form (2.11) by induction on $k \geq 2$. Suppose that (2.11) holds for k . Differentiate this relation with respect to ρ . Each of the summands in $\mathcal{Q}_{ij}^{(k)}$ is a product of factors g^{-1} , g' , and the $\Lambda^{(l)}$ for $1 \leq l \leq k-1$. The derivative of any such factor is again a sum of products of

the same form, except that also $\Lambda^{(k)}$ can appear. In fact, $(g^{-1})' = -g^{-1}g'g^{-1}$, g'' is given by (2.13), and the derivative of a $\Lambda^{(l)}$ by (2.15). Therefore the Leibnitz rule gives a relation of the form (2.11) for $k+1$. This completes the induction and thus also the proof of the first statement of Theorem 1.2.

It is easily seen by induction starting with (2.13) and using (2.15) that for $k \geq 2$, $\mathcal{Q}_{ij}^{(k)}$ has the form

$$(2.16) \quad \mathcal{Q}_{ij}^{(k)} = 2\Lambda_{ij}^{(k-1)} + \overline{\mathcal{Q}}_{ij}^{(k)}(g', \Lambda^{(1)}, \dots, \Lambda^{(k-2)}),$$

where $\overline{\mathcal{Q}}_{ij}^{(k)}$ is a linear combination of partial contractions of the indicated tensors. Thus

$$\mathcal{G}_{ij}^{(k)} = 2\Omega_{ij}^{(k-1)} + \overline{\mathcal{G}}_{ij}^{(k)}(P, \Omega^{(1)}, \dots, \Omega^{(k-2)})$$

for some $\overline{\mathcal{G}}_{ij}^{(k)}$. It follows that

$$(2.17) \quad g^{ij}\partial_\rho^k g_{ij}|_{\rho=0} = g^{ij}\overline{\mathcal{G}}_{ij}^{(k)}(P, \Omega^{(1)}, \dots, \Omega^{(k-2)})$$

for all $k \geq 2$ if n is odd and for $2 \leq k \leq n/2 - 1$ if n is even. However, this reasoning does not apply for $k = n/2$ if n is even, since $\Omega_{ij}^{(n/2-1)}$ is not defined. Nonetheless we claim that this is true also for $k = n/2$, so that

$$(2.18) \quad \mathcal{T}^{(k)} = g^{ij}\overline{\mathcal{G}}_{ij}^{(k)}$$

in Theorem 1.2. To see this, the discussion following (3.16) of [FG2] shows that for n even, $g^{ij}\partial_\rho^{n/2} g_{ij}|_{\rho=0}$ is determined by the condition $\tilde{R}_{\infty\infty} = O(\rho^{n/2-1})$. Now

$$\tilde{R}_{\infty\infty, \underbrace{\infty \dots \infty}_{n/2-2}} = -\tilde{g}^{IJ} \tilde{R}_{\infty IJ\infty, \underbrace{\infty \dots \infty}_{n/2-2}} = -t^{-2} g^{ij} \tilde{R}_{\infty ij\infty, \underbrace{\infty \dots \infty}_{n/2-2}}.$$

Therefore $g^{ij}\tilde{R}_{\infty ij\infty, \underbrace{\infty \dots \infty}_{n/2-2}}|_{\rho=0} = 0$ if $\tilde{R}_{\infty\infty} = O(\rho^{n/2-1})$. Hence $g^{ij}\partial_\rho^{n/2} g_{ij}|_{\rho=0}$ is

determined by requiring $g^{ij}\Lambda_{ij}^{(n/2-1)}|_{\rho=0} = 0$. So setting $k = n/2$ in (2.16), taking the trace, and restricting to $\rho = 0$ proves the second statement of Theorem 1.2 with $\mathcal{T}^{(k)}$ given by (2.18). \square

Equations (2.4) and (2.6) show that

$$\mathcal{G}_{ij}^{(1)} = 2P_{ij}, \quad \mathcal{G}_{ij}^{(2)} = 2\Omega_{ij}^{(1)} + 2P_i^k P_{kj}.$$

Thus $\mathcal{T}^{(2)} = 2P^{ij}P_{ij}$. Formulae for $\mathcal{G}_{ij}^{(k)}$ for $k = 3, 4, 5$ are given in (2.22).

Proof of Corollary 1.3. Taylor expanding the square root of the determinant in (1.2) shows that v_k can be written as a linear combination of complete contractions of the Taylor coefficients $\partial_\rho^l g_{ij}|_{\rho=0}$ for $1 \leq l \leq k-1$ and also $g^{ij}\partial_\rho^k g_{ij}|_{\rho=0}$. (See the end of this section for more details.) Equation (1.6) shows that $\partial_\rho^l g_{ij}|_{\rho=0}$ for

$1 \leq l \leq k-1$ involves only the $\Omega_{ij}^{(s)}$ with $s \leq k-2$, and (2.17) shows that this is also the case for $g^{ij}\partial_\rho^k g_{ij}|_{\rho=0}$. \square

Theorem 1.1 implies that for a fixed background metric g , the equation $v_k(e^{2\omega}g) = c$ is second order in the unknown ω , even though for $k \geq 2$, $v_k(g)$ depends on derivatives of g of order up to $2k-2$. It is possible to say more about the form of $v_k(e^{2\omega}g)$ as a function of ω with g fixed. First we show that $\mathcal{G}_{ij}^{(k)}$ and \mathcal{V}_k have a weighted homogeneity with respect their arguments. Consider a constant rescaling $\hat{g} = s^2g$ with $0 < s \in \mathbb{R}$. The ambient metrics (2.3) are related by the diffeomorphism

$$\hat{t} = ts^{-1}, \quad \hat{x} = x, \quad \hat{\rho} = \rho s^2,$$

with $\hat{g}_\rho = s^2g_\rho$. It follows that $\partial_\rho^k \hat{g}_{ij}|_{\hat{\rho}=0} = s^{2-2k} \partial_\rho^k g_{ij}|_{\rho=0}$. Thus if $\hat{\mathcal{G}}_{ij}^{(k)}$ denotes $\mathcal{G}_{ij}^{(k)}$ evaluated for the metric \hat{g} , then

$$(2.19) \quad \hat{\mathcal{G}}_{ij}^{(k)} = s^{2-2k} \mathcal{G}_{ij}^{(k)}.$$

Suppose a term appears in $\mathcal{G}_{ij}^{(k)}$ whose homogeneity degrees with respect to P , $\Omega^{(1)}, \dots, \Omega^{(k-1)}$ are d_0, d_1, \dots, d_{k-1} , resp., and let $d = \sum_{l=0}^{k-1} d_l$ denote the total degree. Such a term necessarily involves $d-1$ contractions with respect to g^{-1} . By Proposition 2.6, the extended obstruction tensors transform by $\hat{\Omega}^{(l)} = s^{-2l} \Omega^{(l)}$, and of course $\hat{P} = P$ and $\hat{g}^{-1} = s^{-2}g^{-1}$. Thus (2.19) gives

$$-2(d-1) + \sum_{l=1}^{k-1} (-2l)d_l = 2 - 2k,$$

or

$$(2.20) \quad \sum_{l=0}^{k-1} (l+1)d_l = k.$$

This same relation holds for terms appearing in \mathcal{V}_k since $\hat{v}_k = s^{-2k}v_k$ and \mathcal{V}_k involves one more contraction because it is a scalar. Of course, $d_{k-1} = 0$ for \mathcal{V}_k .

Proof of Theorem 1.4. Write $v_k(g) = \mathcal{V}_k(P, \Omega^{(1)}, \dots, \Omega^{(k-2)})$ as a linear combination of complete contractions of P and the $\Omega^{(l)}$ as in Corollary 1.3. The contractions which occur all satisfy (2.20) with $d_{k-1} = 0$. Collect the contractions according to their homogeneity degree $m(=d_0)$ in P : write

$$v_k(g) = \sum_{m=0}^k \mathcal{V}_{k,m}(P, \Omega^{(1)}, \dots, \Omega^{(k-2)}),$$

where $\mathcal{V}_{k,m}$ is the sum of the contractions which are homogeneous of degree m in P . Observe first that $\mathcal{V}_{k,k-1} = 0$ since there are no solutions to (2.20) with $d_0 = k-1$. Next, note that $\mathcal{V}_{k,k}$ depends only on P since $d_0 = k$ in (2.20) forces $d_l = 0$ for $l > 0$. Also, $v_k(g) = \mathcal{V}_{k,k}(P)$ if g is conformally flat, since in this case all $\Omega^{(l)} = 0$.

Since $v_k(g) = \sigma_k(g^{-1}P)$ for g conformally flat, it follows that $\mathcal{V}_{k,k}(P) = \sigma_k(g^{-1}P)$ for general g because any symmetric 2-tensor P_{ij} at a point arises as the Schouten tensor of some conformally flat metric. Thus

$$v_k(g) = \sigma_k(g^{-1}P) + \sum_{m=0}^{k-2} \mathcal{V}_{k,m}(P, \Omega^{(1)}, \dots, \Omega^{(k-2)}),$$

where $\mathcal{V}_{k,m}(P, \Omega^{(1)}, \dots, \Omega^{(k-2)})$ is homogeneous of degree m in P .

Evaluating at \hat{g} gives

$$v_k(\hat{g}) = \sigma_k(\hat{g}^{-1}\hat{P}) + \sum_{m=0}^{k-2} \hat{\mathcal{V}}_{k,m},$$

where $\hat{\mathcal{V}}_{k,m}$ denotes $\mathcal{V}_{k,m}(P, \Omega^{(1)}, \dots, \Omega^{(k-2)})$ evaluated for the metric \hat{g} , i.e. P and the $\Omega^{(l)}$ are replaced by \hat{P} , $\hat{\Omega}^{(l)}$, and the contractions are taken with respect to \hat{g} . Now $\sigma_k(\hat{g}^{-1}\hat{P}) = e^{-2k\omega} \sigma_k(g^{-1}P)$. If we take into account the scaling of v_k and of the $\Omega^{(l)}$ as in the proof of (2.20), it follows that

$$\hat{\mathcal{V}}_{k,m} = e^{-2k\omega} \mathcal{V}_{k,m}(\hat{P}, e^{2\omega} \hat{\Omega}^{(1)}, \dots, e^{2(k-2)\omega} \hat{\Omega}^{(k-2)}),$$

where on the right hand side, $\mathcal{V}_{k,m}(\hat{P}, e^{2\omega} \hat{\Omega}^{(1)}, \dots, e^{2(k-2)\omega} \hat{\Omega}^{(k-2)})$ denotes the sum of the contractions with respect to g of the indicated tensors. Each of the $e^{2l\omega} \hat{\Omega}^{(l)}$ is given by Proposition 2.6, so is a polynomial in $\nabla\omega$ with coefficients depending on g . So if we set

$$r_{k,m}(x, \nabla\omega, \hat{P}) = \mathcal{V}_{k,m}(\hat{P}, e^{2\omega} \hat{\Omega}^{(1)}, \dots, e^{2(k-2)\omega} \hat{\Omega}^{(k-2)}),$$

where the $x, \nabla\omega$ arguments in $r_{k,m}$ correspond to the $e^{2\omega} \hat{\Omega}^{(1)}, \dots, e^{2(k-2)\omega} \hat{\Omega}^{(k-2)}$ arguments in $\mathcal{V}_{k,m}$ and the \hat{P} arguments correspond on both sides, then $r_{k,m}$ is a polynomial in $(\nabla\omega, \hat{P})$ homogeneous of degree m in \hat{P} , with coefficients depending on g . It remains only to bound its degree in $\nabla\omega$.

Consider the expression of $e^{2l\omega} \hat{\Omega}_{ij}^{(l)}$ given by Proposition 2.6. Set $\|0\| = 0$, $\|i\| = 1$ for $1 \leq i \leq n$, $\|\infty\| = 2$, and $\|AB \cdots C\| = \|A\| + \|B\| + \cdots + \|C\|$. Now $p^A_I = 0$ if $\|A\| > \|I\|$ and p^A_I is homogeneous of degree $\|I\| - \|A\|$ in $\nabla\omega$ for $\|I\| \geq \|A\|$. So the term

$$\tilde{R}_{ABCD, F_1 \cdots F_{l-1}}|_{\rho=0, t=1} p^A_\infty p^B_i p^C_j p^D_\infty p^{F_1}_\infty \cdots p^{F_{l-1}}_\infty$$

in Proposition 2.6 is of degree $\leq 2l + 4 - \|ABCD F_1 \cdots F_{l-1}\|$ in $\nabla\omega$. The conformal curvature tensors have the property that $\tilde{R}_{ABCD, F_1 \cdots F_{l-1}}|_{\rho=0, t=1} = 0$ if $\|ABCD F_1 \cdots F_{l-1}\| \leq 3$. This is because in this case at most three of the indices $ABCD F_1 \cdots F_{l-1}$ are nonzero; see Proposition 6.1 of [FG2]. Thus $e^{2l\omega} \hat{\Omega}_{ij}^{(l)}$

is of degree $\leq 2l$ in $\nabla\omega$. If d_1, \dots, d_{k-2} denote the homogeneity degrees with respect to $\Omega_{ij}^{(1)}, \dots, \Omega_{ij}^{(k-2)}$, resp., of a contraction appearing in $\mathcal{V}_{k,m}$, it follows that $r_{k,m}(x, \nabla\omega, \hat{P})$ has degree in $\nabla\omega$ at most

$$\sum_{l=1}^{k-2} 2ld_l = 2(k - m - \sum_{l=1}^{k-2} d_l).$$

Since $d_0 = m < k$, (2.20) shows that $d_l > 0$ for at least one $l \geq 1$, giving the upper bound $2k - 2m - 2$ for the degree of $r_{k,m}$, as claimed. Clearly, for a specific contraction this argument gives a possibly better bound depending on $\sum_{l=1}^{k-2} d_l$. \square

It is possible to derive by hand formulae for some of the extended obstruction tensors and expressions for ambient metric coefficients and renormalized volume coefficients in terms of them. Consider first the extended obstruction tensors. We have already seen that $\Omega_{ij}^{(1)}$ is given by (1.5). Formulae for higher extended obstruction tensors in terms of the Taylor coefficients $\partial_\rho^k g_\rho|_{\rho=0}$ of the ambient metric may be derived inductively starting with (2.12) and using (2.14). For instance, (2.12) together with (2.14) for $k = 0$ give:

$$\tilde{R}_{\infty ij\infty,\infty} = \frac{1}{2}g_{ij}''' - \frac{1}{2}g_{k(i}g_{j)}''^k + \frac{1}{4}g'^{kl}g'_{ik}g'_{jl} - g'_{k(i}\tilde{R}_{j)\infty\infty}^k.$$

g' and g'' at $\rho = 0$ are given by (2.4) and $g'''|_{\rho=0}$ in (3.18) of [FG2]. Substituting these gives

$$\begin{aligned} (n-4)(n-6)\Omega_{ij}^{(2)} &= B_{ij,k}^k - 2W_{kijl}B^{kl} - 4P_k^k B_{ij} \\ &+ (n-4)(4P^{kl}C_{(ij)k,l} - 2C_i^{kl}C_{lj}^k + C_i^{kl}C_{jkl} + 2P_{k,l}^k C_{(ij)}^l - 2W_{kijl}P_m^k P^{ml}). \end{aligned}$$

Carrying out the algorithm by hand to derive the formulae for a few more extended obstruction tensors in terms of the $\partial_\rho^k g_{ij}|_{\rho=0}$ is manageable; this uses only the form (2.3) of the ambient metric. But deriving formulae for $\partial_\rho^k g_{ij}|_{\rho=0}$ in terms of the curvature of the base metric for $k \geq 4$ by solving the Einstein equation is more lengthy.

A similar calculation gives the second Cotton tensor

$$C_{ijl}^{(2)} = \left(2\tilde{R}_{\infty(ij)l,\infty} + \tilde{R}_{\infty ij\infty,l} \right) \Big|_{\rho=0,t=1}.$$

The Bianchi identity allows this to be rewritten as

$$(2.21) \quad C_{ijl}^{(2)} = \left(3\tilde{R}_{\infty ij\infty,l} - \tilde{R}_{\infty li\infty,j} - \tilde{R}_{\infty lj\infty,i} \right) \Big|_{\rho=0,t=1}.$$

The covariant derivative can be evaluated using (2.7) and the formulae for the Christoffel symbols of \tilde{g} given by (3.16) of [FG2] to obtain

$$\tilde{R}_{\infty ij\infty,l}|_{\rho=0,t=1} = \frac{B_{ij,l}}{4-n} - 2P_l^m C_{(ij)m}.$$

Substituting this into (2.21) gives the desired formula for $C_{ijl}^{(2)}$.

The proof of Theorem 1.2 gives the algorithm to make explicit the formulae (1.6) for the ambient metric coefficients in terms of the Schouten tensor and the extended obstruction tensors. This involves the same ingredients as in the derivation of the formulae for the extended obstruction tensors discussed above; it is just a matter of which set of quantities one is solving for inductively in terms of which others. Again, these relations depend only on the form (2.3) of the ambient metric and not on the values of its Taylor coefficients obtained by solving the Einstein equation for \tilde{g} .

Set $g_{ij}^{(k)} = \partial_\rho^k g_{ij}|_{\rho=0}$. We have already seen that

$$\begin{aligned}\frac{1}{2}g_{ij}^{(1)} &= P_{ij} \\ \frac{1}{2}g_{ij}^{(2)} &= \Omega_{ij}^{(1)} + P_i^k P_{jk}.\end{aligned}$$

Carrying out the algorithm of the proof of Theorem 1.2, one obtains:

$$\begin{aligned}(2.22) \quad \frac{1}{2}g_{ij}^{(3)} &= \Omega_{ij}^{(2)} + 4P^k_{(i}\Omega_{j)k}^{(1)} \\ \frac{1}{2}g_{ij}^{(4)} &= \Omega_{ij}^{(3)} + 6P^k_{(i}\Omega_{j)k}^{(2)} + 4\Omega^{(1)k}_i\Omega_{jk}^{(1)} + 4P^k_i P^l_j \Omega_{kl}^{(1)} \\ \frac{1}{2}g_{ij}^{(5)} &= \Omega_{ij}^{(4)} + 8P^k_{(i}\Omega_{j)k}^{(3)} + 14\Omega^{(2)k}_{(i}\Omega_{j)k}^{(1)} + 10P^k_i P^l_j \Omega_{kl}^{(2)} + 16P^k_{(i}\Omega_{j)k}^{(1)l}\Omega_{kl}^{(1)}.\end{aligned}$$

The algorithm also easily gives the leading terms: for $k \geq 3$ one has

$$\frac{1}{2}g_{ij}^{(k)} = \Omega_{ij}^{(k-1)} + 2(k-1)P^l_{(i}\Omega_{j)l}^{(k-2)} + \mathcal{J}_{ij}^{(k)},$$

where $\mathcal{J}_{ij}^{(k)}$ is a linear combination of contractions of the tensors $P, \Omega^{(1)}, \dots, \Omega^{(k-3)}$ satisfying (2.20).

Finally, the renormalized volume coefficients can be calculated from the ambient metric coefficients by expanding the volume form. Set $D = \det g_\rho / \det g_0$. Then $D' = Dg^{ij}g'_{ij}$. Successive differentiation of this relation gives formulae for $\partial_\rho^k D/D$ in terms of g^{-1} and derivatives of g . For example,

$$D'' = Dg^{ij}g''_{ij} - Dg^{ik}g^{jl}g'_{kl}g'_{ij} + D(g^{ij}g'_{ij})^2.$$

The Taylor coefficients of D are then obtained by evaluating these relations at $\rho = 0$ and substituting the above formulae for the Taylor coefficients of g_ρ . Composing with the Taylor expansion of \sqrt{x} about $x = 1$ gives the v_k according to (1.2). It is straightforward but tedious to carry this out. The result for the first few v_k is:

$$\begin{aligned}(2.23) \quad v_1 &= \sigma_1 \\ v_2 &= \sigma_2 \\ v_3 &= \sigma_3 - \frac{1}{3} \operatorname{tr} (P\Omega^{(1)}) \\ v_4 &= \sigma_4 + \frac{1}{3} \operatorname{tr} (P^2\Omega^{(1)}) - \frac{1}{3} (\operatorname{tr} P) \operatorname{tr} (P\Omega^{(1)}) - \frac{1}{12} \operatorname{tr} (P\Omega^{(2)}) - \frac{1}{12} \operatorname{tr} (\Omega^{(1)})^2.\end{aligned}$$

Here we have omitted the argument $g^{-1}P$ of the σ_k . Also omitted are the g^{-1} factors raising the indices in the trace terms. These σ_k are given by:

$$\begin{aligned}\sigma_1 &= J \\ \sigma_2 &= \frac{1}{2} (J^2 - \operatorname{tr} P^2) \\ \sigma_3 &= \frac{1}{6} (2 \operatorname{tr} P^3 - 3J \operatorname{tr} P^2 + J^3) \\ \sigma_4 &= \frac{1}{24} (-6 \operatorname{tr} P^4 + 8J \operatorname{tr} P^3 + 3(\operatorname{tr} P^2)^2 - 6J^2 \operatorname{tr} P^2 + J^4),\end{aligned}$$

where $J = \operatorname{tr} P = R/2(n-1)$.

3. LINEARIZATION

Let X be a manifold-with-boundary and set $\partial X = M$. If $[g]$ is a conformal class of metrics of signature (p, q) on M , recall that a metric g_+ of signature $(p+1, q)$ on X° is said to be conformally compact with conformal infinity $(M, [g])$ if $u^2 g_+$ extends smoothly to X with $u^2 g_+|_M$ nondegenerate and $u^2 g_+|_{TM} \in [g]$, where u is a defining function for M . The function $|du|_{u^2 g_+}^2|_M$ is independent of the choice of u ; g_+ is said to be asymptotically hyperbolic if $|du|_{u^2 g_+}^2|_M = 1$.

Let g_+ be asymptotically hyperbolic and let g be a choice of metric in the conformal class on M . Then there is an open neighborhood of $M (= M \times \{0\})$ in $M \times [0, \infty)$ on which there is a unique diffeomorphism φ to a neighborhood of M in X such that $\varphi|_M$ is the identity, and such that $\varphi^* g_+$ takes the form

$$\varphi^* g_+ = r^{-2} (dr^2 + h(r)),$$

where $h(r)$ (denoted h_r previously) is a 1-parameter family of metrics on M of signature (p, q) satisfying $h(0) = g$. Here r denotes the variable in $[0, \infty)$. See §5 of [GL].

Suppose we choose a conformally related metric $\hat{g} = e^{2\omega} g$, where $\omega \in C^\infty(M)$. Then \hat{g} induces another diffeomorphism $\hat{\varphi}$ from a neighborhood of M in $M \times [0, \infty)$ to a neighborhood of M in X such that $\hat{\varphi}^* g_+ = r^{-2} (dr^2 + \hat{h}(r))$, where $\hat{h}(r)$ is a 1-parameter family of metrics on M , satisfying $\hat{h}(0) = \hat{g}$, uniquely determined by g_+ , g , and ω . Consider the infinitesimal dependence of $\hat{h}(r)$ on ω . For each t , denote by $\hat{h}^t(r)$ the 1-parameter family of metrics obtained from the conformal representative $\hat{g}^t = e^{2t\omega} g$. Let $\delta = \partial_t|_{t=0}$ denote the operation of taking the infinitesimal conformal variation. For example,

$$\delta h(r) = \partial_t \hat{h}^t(r)|_{t=0}.$$

We sometimes suppress writing the argument for $h(r)$ and $\delta h(r)$; the r dependence of h and δh is to be understood.

Theorem 3.1. *Under infinitesimal conformal change of g , $h(r)$ transforms by:*

$$(3.1) \quad (\delta h)_{ij} = \omega(2 - r \partial_r) h_{ij} + 2 \nabla_{(i} X_{j)},$$

where X^i is the r -dependent family of vector fields on M given by

$$(3.2) \quad X^i(r) = \int_0^r s h^{ij}(s) ds \partial_j \omega.$$

Here $X_j(r) = h_{ij}(r)X^i(r)$, and ∇_i denotes the covariant derivative on M with respect to $h(r)$ with r fixed.

Note in (3.2) that $\partial_j \omega$ is independent of r . An immediate consequence of Theorem 3.1 is the fact that for each r , the transformation rule for infinitesimal conformal change of $h(r)$ involves at most second derivatives of ω .

Proof. For each t , we have a diffeomorphism φ_t such that $\varphi_t|_M$ is the identity and $\varphi_t^* g_+ = r^{-2} (dr^2 + \widehat{h}^t(r))$. So

$$(\varphi_0^{-1} \circ \varphi_t)^* \left(\frac{dr^2 + h(r)}{r^2} \right) = \frac{dr^2 + \widehat{h}^t(r)}{r^2}.$$

Differentiate with respect to t at $t = 0$ to deduce that there is a vector field X near M in $M \times [0, \infty)$ such that $X|_M = 0$ and

$$\mathcal{L}_X \left(\frac{dr^2 + h}{r^2} \right) = \frac{\delta h}{r^2},$$

where \mathcal{L} denotes the Lie derivative. Expanding the left hand side and then multiplying by r^2 gives

$$(3.3) \quad -2r^{-1}X(r)(dr^2 + h) + \mathcal{L}_X(dr^2) + \mathcal{L}_X h = \delta h.$$

Now write $X = X^0 \partial_r + X^i \partial_i$. Then

$$\begin{aligned} X(r) &= X^0 \\ \mathcal{L}_X(dr^2) &= 2dX^0 dr = 2\partial_r X^0 dr^2 + 2\partial_i X^0 dx^i dr \\ \mathcal{L}_X h &= (2\nabla_{(i} X_{j)} + X^0 \partial_r h_{ij}) dx^i dx^j + 2h_{ij} \partial_r X^j dr dx^i, \end{aligned}$$

where $X_j = h_{jk} X^k$ and ∇_i is the covariant derivative on M with respect to $h(r)$ with r fixed. Substitute these into (3.3) and then equate the coefficients of dr^2 , $dr dx^i$ and $dx^i dx^j$ on the two sides of (3.3). One obtains

$$\begin{aligned} (3.4) \quad & -2r^{-1}X^0 + 2\partial_r X^0 = 0 \\ & 2\partial_i X^0 + 2h_{ij} \partial_r X^j = 0 \\ & -2r^{-1}X^0 h_{ij} + 2\nabla_{(i} X_{j)} + X^0 \partial_r h_{ij} = \delta h_{ij}. \end{aligned}$$

The first equation shows that $X^0 = cr$ where c is independent of r , i.e. c is just a function of $x \in M$. Substitute this into the last equation and evaluate at $r = 0$.

Recalling that $X = 0$ at $r = 0$ and $\delta h = 2\omega h = 2\omega g$ at $r = 0$, one obtains $c = -\omega$. So now we know $X^0 = -\omega r$. Substitute this into the second equation to obtain

$$\partial_r X^i = r h^{ij} \partial_j \omega.$$

Now integrate in r to solve for X^i ; $\partial_j \omega$ is a constant with respect to the integration. Using the initial condition $X^i = 0$ at $r = 0$, one obtains

$$X^i = \int_0^r s h^{ij}(s) ds \partial_j \omega.$$

Substituting $X^0 = -\omega r$ into the third line of (3.4) gives (3.1). \square

It is useful to introduce the new variable $\rho = -\frac{1}{2}r^2$ as in §2 in the infinitesimal transformation law (3.1). Set $g(\rho) = h(r)$ (denoted g_ρ previously) and $Y^i(\rho) = X^i(r)$. Then (3.1), (3.2) become (1.8), (1.9).

Consider now the case where $g_+ = r^{-2}(dr^2 + h(r))$ is a Poincaré metric whose Taylor expansion (to the appropriate order for n even) is determined along M by the choice of an initial metric g via the Einstein equation $\text{Ric}(g_+) = -ng_+$. The Taylor coefficients of $g(\rho)$ are the natural tensors studied in §2, so the Taylor expansion of (1.8) gives the infinitesimal transformation laws of these tensors. For example, (1.9) shows that $\partial_\rho Y^i|_{\rho=0} = -g^{ij} \partial_j \omega$, so differentiating (1.8) once at $\rho = 0$ and recalling (2.4) recovers the infinitesimal transformation law $\delta P_{ij} = -\omega_{ij}$ of the Schouten tensor. In general, in (1.8) the term $2\omega(1 - \rho \partial_\rho)g_{ij}$ encodes the scaling of each coefficient and the term $2\nabla_{(i} Y_{j)}$ carries the dependence on derivatives of ω . It follows that the infinitesimal transformation laws of all these natural tensors (subject to the usual truncation for n even) involves at most second derivatives of ω .

An easy consequence of Theorem 3.1 is a similar formula for the infinitesimal transformation laws of the renormalized volume coefficients. First suppose that g_+ is asymptotically hyperbolic with conformal infinity $(M, [g])$ but not necessarily asymptotically Einstein, as in the setting of Theorem 3.1. Define $v(\rho)$ by (1.10).

Proposition 3.2. *Under infinitesimal conformal change of $g(0)$, $v(\rho)$ transforms by:*

$$\begin{aligned} \delta v &= -2\omega \rho \partial_\rho v + v \nabla_i^{(\rho)} Y^i \\ (3.5) \quad &= -2\omega \rho \partial_\rho v + \nabla_i^{(0)} (v Y^i), \end{aligned}$$

where Y^i is given by (1.9) and $\nabla_i^{(\rho)}$ is the covariant derivative with respect to $g(\rho)$ with ρ fixed.

Proof. Under a conformal transformation, $\delta \det g(0) = 2n\omega \det g(0)$, so

$$\frac{\delta v}{v} = \delta \log v = \frac{1}{2} (g^{ij} \delta g_{ij} - 2n\omega).$$

Substitution of (1.8) gives

$$\frac{\delta v}{v} = -\omega g^{ij} \rho \partial_\rho g_{ij} + g^{ij} \nabla_i^{(\rho)} Y_j = -2\omega \frac{\rho \partial_\rho v}{v} + \nabla_i^{(\rho)} Y^i.$$

This gives the first line of (3.5). The second line follows since $v \nabla_i^{(\rho)} Y^i = \nabla_i^{(0)} (v Y^i)$ for any vector field Y^i . \square

Proof of Theorem 1.5. Take g_+ in Proposition 3.2 to be an asymptotically Einstein metric whose Taylor expansion is determined by $g = g(0)$. By (1.2), the coefficient of ρ^k in δv is δv_k . So taking Taylor coefficients in (3.5) and recalling (1.9) gives (1.11) with $L_{ij}^{(k)}$ given by the first equality of (1.12). The second equality of (1.12) follows upon expanding via the Leibnitz rule. \square

The fact that the second term on the right hand side of (3.5) is a divergence implies that it drops out when considering the infinitesimal conformal change of the volume of M relative to the metrics $g(\rho)$. Suppose M is compact and set

$$V(\rho) \equiv \text{Vol}_{g(\rho)}(M) = \int_M v(\rho) dv_{g(0)}.$$

Integration of (3.5) gives for each ρ :

$$(3.6) \quad \delta V = \int_M \delta v dv_{g(0)} = -2 \int_M \omega \rho \partial_\rho v dv_{g(0)}.$$

Taking Taylor coefficients in (3.6) (or integrating (1.11) over M), it follows that:

Proposition 3.3. *Suppose $k \geq 1$ with $k \leq n/2$ if n is even, and suppose M is compact. Then*

$$\int_M \delta v_k dv_g = -2k \int_M v_k \omega dv_g.$$

Proposition 3.3 is the main ingredient in the proof of the result of Chang-Fang [CF]. Consider the functionals

$$\mathcal{F}_k(g) = \int_M v_k(g) dv_g$$

as g varies over a conformal class. \mathcal{F}_k is defined for all $k \geq 1$ if k is odd and for $1 \leq k \leq n/2$ if n is even. It was shown in [G] that if n is even, then $\mathcal{F}_{n/2}(g)$ is conformally invariant, i.e. is constant on each conformal class. The Chang-Fang theorem gives the constrained Euler-Lagrange equation for the other values of k :

Theorem 3.4. *Suppose $k \geq 1$ and $k < n/2$ if n is even. The Euler-Lagrange equation for $\mathcal{F}_k(g)$ as g varies over a conformal class, subject to the constraint $\text{Vol}_g(M) = 1$, is $v_k(g) = c$.*

Proof. The constrained Euler-Lagrange equation for \mathcal{F}_k is obtained by requiring that $\mathcal{F}_k - \lambda \text{Vol}(M)$ vanishes to first order in ω under a conformal change $\hat{g} = e^{2\omega}g$, where λ is a Lagrange multiplier. This is therefore the condition

$$(3.7) \quad \delta(\mathcal{F}_k - \lambda \text{Vol}(M))(g) = 0 \quad \text{for all } \omega.$$

Proposition 3.3 together with the fact that $\delta dv_g = n\omega dv_g$ give

$$\delta\mathcal{F}_k(g) = \int_M (\delta v_k dv_g + v_k \delta dv_g) = (n - 2k) \int_M v_k \omega dv_g.$$

If $n = 2k$ we recover the conformal invariance of $\mathcal{F}_{n/2}$. Otherwise (3.7) becomes

$$(n - 2k) \int_M v_k \omega dv_g - n\lambda \int_M \omega dv_g = 0 \quad \text{for all } \omega,$$

which gives $v_k(g) = n\lambda/(n - 2k) = c$. \square

Thus if we fix a background metric g in the conformal class, then the critical points of $\mathcal{F}_k(e^{2\omega}g)$ as a function of ω are those ω for which $v_k(e^{2\omega}g) = c$. So we recover the second order fully nonlinear operator whose structure was studied in §2. Its linearization at $\omega = 0$ is of course just δv_k , so Theorem 1.5 gives:

Proposition 3.5. *Suppose $k \geq 1$ with $k \leq n/2$ if n is even. Let $\mathcal{P}_k(\omega)$ denote the linearization at $\omega = 0$ of the operator $\omega \rightarrow v_k(e^{2\omega}g)$ with g fixed. Then*

$$\mathcal{P}_k(\omega) = \nabla_i \left(L_{(k)}^{ij} \nabla_j \omega \right) - 2k v_k \omega,$$

with $L_{(k)}^{ij}$ given by (1.12).

In considering (1.12), recall that $g_{ij}(\rho)$ is the series determined by $g = g(0)$ upon formally solving the Einstein equation $\text{Ric}(\tilde{g}) = 0$. Hence identification of $\partial_\rho^{l-1} g^{ij}|_{\rho=0}$ requires knowledge of the coefficients $\partial_\rho^m g_{ij}|_{\rho=0}$ in the ambient metric expansion as well as calculation of the Taylor coefficients of the inverse in terms of these. In any case, it is clear from Theorem 1.2 and Corollary 1.3 that each $L_{(k)}^{ij}$ (with $k \leq n/2$ for n even) is a natural tensor which can be written as a linear combination of contractions of the Schouten tensor and the extended obstruction tensors with coefficients independent of the dimension.

For small k , it is possible to calculate $L_{(k)}^{ij}$ from (1.12) using (2.22) and (2.23). Alternately, one can simply linearize the explicit expressions (2.23). In the latter approach, one uses from Proposition 3.5 that \mathcal{P}_k is determined once one knows its principal part $L_{(k)}^{ij}$. Thus it suffices to calculate the principal part of the linearization from (2.23). So in linearizing (2.23), one can ignore contributions from the $\Omega^{(l)}$ and simply apply the Leibnitz rule to P_{ij} and use $\delta P_{ij} = -\omega_{ij}$.

Recall that the linearization of σ_k can be expressed in terms of its polarization. If $A_j^i(t)$ is a 1-parameter family of endomorphisms of a vector space, then the

relation

$$(3.8) \quad \sigma_k(A)^\cdot = \text{tr} \left(T_{(k-1)}(A) \dot{A} \right)$$

defines an endomorphism-valued polynomial $T_{(k-1)}(A)$ homogeneous of degree $k-1$ in A . Let $\bar{\sigma}_k$ denote the symmetric k -linear form obtained by complete polarization of σ_k , i.e. $\bar{\sigma}_k(A_1, \dots, A_k)$ is linear in each A_l , symmetric, and satisfies $\bar{\sigma}_k(A, \dots, A) = \sigma_k(A)$. Then the Leibnitz rule shows that

$$\text{tr} \left(T_{(k-1)}(A) B \right) = k \bar{\sigma}_k(A, \dots, A, B).$$

In the sequel, our vector space will be equipped with a non-degenerate quadratic form which we use to raise and lower indices, and A_{ij} will be symmetric. So we will usually write (3.8) in the form

$$\sigma_k(A)^\cdot = T_{(k-1)}^{ij}(A) \dot{A}_{ij}.$$

The homogeneity of σ_k together with $\delta P_{ij} = -\omega_{ij}$ give

$$(3.9) \quad \delta \left(\sigma_k(g^{-1}P) \right) = -T_{(k-1)}^{ij}(g^{-1}P) \omega_{ij} - 2k \sigma_k(g^{-1}P) \omega,$$

so that the principal part of the linearization of $\sigma_k(g^{-1}P)$ is $-T_{(k-1)}^{ij}(g^{-1}P)$. In the following, we suppress writing the argument $g^{-1}P$ of $T_{(k-1)}^{ij}$ and we write $\Omega_{(l)}$ instead of $\Omega^{(l)}$. Either directly linearizing (2.23) or calculating from (1.12), one obtains:

$$\begin{aligned} L_{(1)}^{ij} &= -g^{ij} \\ L_{(2)}^{ij} &= -T_{(1)}^{ij} \\ L_{(3)}^{ij} &= -T_{(2)}^{ij} + \frac{1}{3} \Omega_{(1)}^{ij} \\ L_{(4)}^{ij} &= -T_{(3)}^{ij} - \frac{2}{3} P_k^{(i} \Omega_{(1)}^{j)k} + \frac{1}{3} P_{kl} \Omega_{(1)}^{kl} g^{ij} + \frac{1}{3} P_k^k \Omega_{(1)}^{ij} + \frac{1}{12} \Omega_{(2)}^{ij}. \end{aligned}$$

Recall from the discussion in §2 that if g is locally conformally flat, then the v_k are defined for all k also for n even, and $v_k(g) = \sigma_k(g^{-1}P)$ for all k in all dimensions. The invariance of the ambient metric holds to all orders in all dimensions and this was the fundamental ingredient used in the proof of Theorem 3.1. Thus all the arguments and results of this section apply without the restriction $k \leq n/2$ for n even if g is locally conformally flat. In particular, this gives another argument for the variational character of the σ_k in this case.

The relations asserted by Theorem 3.1, Proposition 3.2 and Theorem 1.5 are not obvious for locally conformally flat metrics. Equation (2.1) can be written

$$(3.10) \quad g_{ij}(\rho) = g_{ij}(0) + 2P_{ij}\rho + P_{ik}P^k_{j}\rho^2.$$

Let us set $\gamma = g(0)$ and $A = \gamma^{-1}P$. Then (3.10) can be written $g(\rho) = \gamma(I + \rho A)^2$. Therefore

$$(3.11) \quad \int_0^\rho g^{-1}(u) du = \int_0^\rho (I + uA)^{-2} du \gamma^{-1} = \rho(I + \rho A)^{-1} \gamma^{-1}.$$

Hence

$$g(\rho) \int_0^\rho g^{-1}(u) du = \rho \gamma (I + \rho A) \gamma^{-1}.$$

Comparing with (1.9) gives

$$(3.12) \quad Y_j = -\rho (\delta_j^k + \rho P_j^k) \omega_k.$$

Taking the conformal variation in (3.10) yields

$$(\delta g)_{ij} = 2\omega \gamma_{ij} - 2\omega_{ij}\rho + (-2\omega P_{ik}P_j^k - 2\omega_{k(i}P_{j)}^k) \rho^2.$$

The fact that this agrees with the right hand side of (1.8) can be verified using (3.12) and the relation between the Levi-Civita connections of $g(\rho)$ and γ derived in Lemma 7.3 of [FG2].

Similarly, it can be verified directly that if g is locally conformally flat, then (1.12) reduces to $L_{(k)}^{ij} = -T_{(k-1)}^{ij}(g^{-1}P)$ for $k \leq n$ and to 0 for $k > n$, and that (1.11) reduces to (3.9). The identification of the $L_{(k)}^{ij}$ is clearly equivalent to showing that

$$v(\rho) \int_0^\rho g^{ij}(u) du = \sum_{k=1}^n T_{(k-1)}^{ij}(g^{-1}P) \rho^k.$$

Now $v(\rho) = \det(I + \rho A)$, so (3.11) shows that this can be rewritten as

$$\det(I + \rho A)(I + \rho A)^{-1} = \sum_{k=0}^{n-1} T_{(k)}(A) \rho^k.$$

It is standard and can be seen in a variety of ways that this is a reformulation of the definition of the $T_{(k)}(A)$. Thus one concludes that $L_{(k)}^{ij} = -T_{(k-1)}^{ij}(g^{-1}P)$.

Finally, (1.11) reduces to (3.9) since $\nabla_i (T_{(k-1)}^{ij}(g^{-1}P)) = 0$. The fact that $\nabla_i (T_{(k-1)}^{ij}(g^{-1}P)) = 0$ for locally conformally flat metrics follows from the vanishing of the Cotton tensor and is essentially equivalent to the variational characterization of the σ_k . See [V] or [BG].

Theorem 3.1 can be used as the basis for another proof of Theorem 1.1, the fact that under conformal change, the ambient metric coefficients $\partial_\rho^k g_{ij}|_{\rho=0}$ and the v_k depend on at most second derivatives of ω . As observed above, the fact that this is true under infinitesimal conformal change is immediate from Theorem 3.1. Thus Theorem 1.1 follows if we can prove that the full conformal transformation law depends on at most 2 derivatives of the conformal factor, assuming that this is the

case for the infinitesimal transformation law. We formulate a general result along these lines.

Consider a polynomial natural tensor $T(g)$ depending on a Riemannian metric in dimension $n \geq 2$, of contravariant rank a and covariant rank b . $T(g)$ may be expressed by evaluating a linear combination of partial contractions of covariant indices against contravariant indices of g , g^{-1} , and the covariant derivatives $\nabla^r R$, $r \geq 0$, of the curvature tensor R of g . Each such partial contraction can be written in the form

$$(3.13) \quad \text{pcontr} \left(\nabla^{r_1} R \otimes \cdots \otimes \nabla^{r_M} R \otimes g \otimes \cdots \otimes g \otimes g^{-1} \otimes \cdots \otimes g^{-1} \right).$$

Our convention is that the curvature tensor R has contravariant rank 0 and covariant rank 4.

We say that T has homogeneity $h \in \mathbb{R}$ if

$$T(e^{2\omega} g) = e^{h\omega} T(g), \quad \omega \in \mathbb{R}.$$

We assume throughout that T has a well-defined homogeneity; this is no loss of generality since a general natural tensor is the sum of its homogeneous parts in this sense and all of our considerations respect homogeneity. If the contraction (3.13) has P factors of g , Q factors of g^{-1} , and involves C contractions of a covariant index against a contravariant index, then the contravariant rank a , covariant rank b , and homogeneity h of the resulting tensor are given by

$$\begin{aligned} a &= 2Q - C \\ b &= 4M + \sum_{i=1}^M r_i + 2P - C \\ h &= 2(M + P - Q). \end{aligned}$$

In particular, the quantity

$$L \equiv 2M + \sum r_i = b - a - h$$

is determined just by the rank and homogeneity of T . L is called the *level* of T ; it is the total number of derivatives of g occurring in T . Clearly $L \geq 0$.

If T has homogeneity h , the full conformal variation $\Delta T(g, \omega)$ of T is defined to be

$$\Delta T(g, \omega) \equiv e^{-h\omega} T(e^{2\omega} g) - T(g)$$

for smooth ω . Then $\Delta T(g, \omega)$ is a natural tensor depending on g and the scalar function ω . It can be obtained by evaluating a linear combination of partial contractions of g , g^{-1} , the $\nabla^r R$ for $r \geq 0$, and the covariant derivatives $\nabla^l \omega$, $l \geq 1$, of ω with respect to the Levi-Civita connection of g , each contraction of which contains at least one of the $\nabla^l \omega$. In this discussion we use a slightly modified

infinitesimal conformal variation operator $\underline{\delta}$ by subtracting the scaling term from δ . If T is a natural tensor of homogeneity h , define

$$\underline{\delta}T(g, \omega) \equiv \frac{d}{dt} e^{-ht\omega} T(e^{2t\omega} g) \Big|_{t=0} = \delta T - h\omega T.$$

Then $\underline{\delta}T(g, \omega)$ is a natural tensor depending on g and ω ; it is obtained from $\Delta T(g, \omega)$ by keeping only the terms which are linear in the derivatives of ω . It is evident that when viewed as a function of g , $\Delta T(g, \omega)$ and $\underline{\delta}T(g, \omega)$ also have homogeneity h :

$$\Delta T(e^{2\Upsilon} g, \omega) = e^{h\Upsilon} \Delta T(g, \omega), \quad \underline{\delta}T(e^{2\Upsilon} g, \omega) = e^{h\Upsilon} \underline{\delta}T(g, \omega), \quad \Upsilon \in \mathbb{R}.$$

We may consider the infinitesimal conformal variation of $\underline{\delta}T$ in g :

$$\underline{\delta}^2 T(g, \omega, \Upsilon) \equiv \frac{d}{dt} e^{-ht\Upsilon} \underline{\delta}T(e^{2t\Upsilon} g, \omega) \Big|_{t=0}.$$

The equality of second mixed partials implies that

$$(3.14) \quad \underline{\delta}^2 T(g, \omega, \Upsilon) = \underline{\delta}^2 T(g, \Upsilon, \omega).$$

Let $U(g, \omega)$ be a polynomial natural tensor depending on g and ω (for example $U = \underline{\delta}T$ or $U = \Delta T$). For $m \geq 0$, we will say that U involves at most m derivatives of ω if it can be obtained by evaluating a linear combination of partial contractions in which only the tensors $\nabla^l \omega$, $1 \leq l \leq m$ appear, together with g , g^{-1} , and the $\nabla^r R$ for $r \geq 0$.

Proposition 3.6. *Let $m \geq 0$. If $\underline{\delta}T$ involves at most m derivatives of ω , then the same is true for ΔT .*

The case $m = 0$ is the well-known statement that if a natural tensor is infinitesimally conformally invariant, then it is conformally invariant. The proof in this case is simpler than in the case $m > 0$. Clearly $\underline{\delta}T$ involves at most m derivatives of ω if and only if δT involves at most m derivatives of ω .

Proof. The proof is by induction on the level $L = b - a - h$ of T . First consider the case $L = 0$. If a contraction (3.13) appears in an expression for T , then the relation $L = 2M + \sum r_i$ forces $M = 0$. Thus T is a linear combination of partial contractions only of g and g^{-1} . Such a T is conformally invariant, so the desired conclusion is automatic.

Assume now that the result is true for natural tensors whose level L satisfies $L \leq N$ for some $N \geq 0$. Suppose $T(g)$ is a natural tensor of some homogeneity h and level $N + 1$, for which $\underline{\delta}T(g, \omega)$ involves at most m derivatives of ω . We can write

$$(3.15) \quad \underline{\delta}T(g, \omega) = \omega_{i_1} S_{(1)}^{i_1}(g) + \omega_{i_1 i_2} S_{(2)}^{i_1 i_2}(g) + \cdots + \omega_{i_1 \dots i_m} S_{(m)}^{i_1 \dots i_m}(g),$$

where for $1 \leq l \leq m$, $S_{(l)}(g)$ is a natural tensor of homogeneity h whose covariant rank equals that of T and whose contravariant rank is l more than that of T . Here

$\omega_{i_1 \dots i_l}$ denotes the components of $\nabla^l \omega$. Since the level of T is $N+1$, it follows that the level of $S_{(l)}$ is $N+1-l \leq N$. Since for each l , the skew-symmetrization of $\nabla^l \omega$ in any two indices can be expressed by the Ricci identity in terms of the tensors $\nabla^j \omega$ with $1 \leq j \leq l-2$ and the $\nabla^r R$, it follows inductively that $S_{(l)}(g)$ can be taken to be symmetric in the indices $i_1 \dots i_l$. Under this condition the $S_{(l)}(g)$ are uniquely determined.

We claim that each of the $\underline{\delta} S_{(l)}(g, \omega)$ involves at most m derivatives of ω . To see this, take the infinitesimal conformal variation of (3.15) in g with respect to a conformal change $\widehat{g}_t = e^{2t\Upsilon} g$. The infinitesimal conformal variation of the right hand side may be calculated via the Leibnitz rule. Each of the terms $\nabla^l \omega$ has a variation corresponding to the change of the connection. It is clear that $\underline{\delta}(\nabla^l \omega)$ involves at most l derivatives of ω and Υ . Thus it follows that

$$\begin{aligned} \underline{\delta}^2 T(g, \omega, \Upsilon) \\ = \omega_{i_1} \underline{\delta} S_{(1)}^{i_1}(g, \Upsilon) + \omega_{i_1 i_2} \underline{\delta} S_{(2)}^{i_1 i_2}(g, \Upsilon) + \dots + \omega_{i_1 \dots i_m} \underline{\delta} S_{(m)}^{i_1 \dots i_m}(g, \Upsilon) + U(g, \omega, \Upsilon), \end{aligned}$$

where $U(g, \omega, \Upsilon)$ is a natural tensor depending on g , ω , and Υ which involves at most m derivatives of ω and Υ . Using (3.14), we obtain

$$\begin{aligned} \omega_{i_1} \underline{\delta} S_{(1)}^{i_1}(g, \Upsilon) + \omega_{i_1 i_2} \underline{\delta} S_{(2)}^{i_1 i_2}(g, \Upsilon) + \dots + \omega_{i_1 \dots i_m} \underline{\delta} S_{(m)}^{i_1 \dots i_m}(g, \Upsilon) + \widetilde{U}(g, \omega, \Upsilon) \\ = \Upsilon_{i_1} \underline{\delta} S_{(1)}^{i_1}(g, \omega) + \Upsilon_{i_1 i_2} \underline{\delta} S_{(2)}^{i_1 i_2}(g, \omega) + \dots + \Upsilon_{i_1 \dots i_m} \underline{\delta} S_{(m)}^{i_1 \dots i_m}(g, \omega), \end{aligned}$$

where again \widetilde{U} involves at most m derivatives of ω and Υ . Since the left hand side involves at most m derivatives of ω , the same is true of the right hand side. Therefore this must also hold for the coefficient of each of the $\Upsilon_{i_1 \dots i_l}$. Hence each of the $\underline{\delta} S_{(l)}(g, \omega)$ involves at most m derivatives of ω as claimed. Thus the induction hypothesis applies to each of the $S_{(l)}(g)$, and we deduce that for $1 \leq l \leq m$, $\Delta S_{(l)}(g, \omega)$ involves at most m derivatives of ω .

Next, recall that ΔT can be recovered by integrating $\underline{\delta} T$. To see this, note first that

$$\begin{aligned} \frac{d}{dt} (e^{-ht\omega} T(e^{2t\omega} g)) &= \frac{d}{ds} (e^{-h(t+s)\omega} T(e^{2(t+s)\omega} g)) \Big|_{s=0} \\ &= e^{-ht\omega} \frac{d}{ds} (e^{-hs\omega} T(e^{2s\omega} e^{2t\omega} g)) \Big|_{s=0} = e^{-ht\omega} \underline{\delta} T(e^{2t\omega} g, \omega). \end{aligned}$$

Thus

$$\begin{aligned} \Delta T(g, \omega) &= e^{-h\omega} T(e^{2\omega} g) - T(g) = \int_0^1 \frac{d}{dt} (e^{-ht\omega} T(e^{2t\omega} g)) dt \\ (3.16) \quad &= \int_0^1 e^{-ht\omega} \underline{\delta} T(e^{2t\omega} g, \omega) dt. \end{aligned}$$

Apply (3.15) to evaluate $\underline{\delta} T(e^{2t\omega} g, \omega)$. The occurrences of $\nabla^l \omega$, $1 \leq l \leq m$ on the right hand side of (3.15) now have to be evaluated using the Levi-Civita connection

of $e^{2t\omega}g$. It is clear that for fixed t , each such evaluation gives rise to a natural tensor depending on g and ω which involves at most m derivatives of ω . Likewise, for each t we have

$$e^{-ht\omega}S_{(l)}(e^{2t\omega}g) = S_{(l)}(g) + \Delta S_{(l)}(g, t\omega),$$

and the right hand side is a family parametrized by t of natural tensors depending on g and ω which involves at most m derivatives of ω . Substituting into (3.16) and integrating in t , it follows that $\Delta T(g, \omega)$ involves at most m derivatives of ω . This completes the induction step. \square

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